THE INDUCTION PRINCIPLE

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1. *The Fibonacci sequence is defined by* $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, n \geq 0$. *Prove the following properties related to it:*

- $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, *where* $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$.
- $F - n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots$.
- $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$.
- $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$.
- $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.
- $F_1 + F_3 + \cdots + F_{2n+1} = F_{2n+2}, 1 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1}$.
- $F_nF_{n+1} - F_{n-2}F_{n-1} = F_{2n-1}, F_{n+1}F_{n+2} - F_nF_{n+3} = (-1)^n$.
- $F_{n-1}^2 + F_n^2 = F_{2n-1}, F_n^2 + 2F_{n-1}F_n = F_{2n}, F_n(F_{n+1} + F_{n-1}) = F_{2n}$.
- $F_1F_2 + F_2F_3 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$.
- $F_n^3 + F_{n+1}^3 - F_{n-1}^3 = F_{3n}$.
- *If* $m \mid n$, *then* $F_m \mid F_n$.
- $(F_m, F_n) = F_{(m,n)}$.
- *Let* t *be the positive root of* $t^2 = t + 1$, *then* $t = 1 + \frac{1}{t}$, *from which follows the continued fraction expansion*

  $$t = 1 + \cfrac{1}{1 + \cfrac{1}{!+ \cfrac{1}{1+ \cfrac{1}{1+ \cfrac{1}{1+\cdots}}}}}$$

  with the convergents

  $$t_1 = 1, t_2 = 1 + \frac{1}{1}, \ldots$$

  then, $t_n = \frac{F_{n+1}}{F_n}$.
- $\sum_{i=1}^{\infty} \frac{1}{F_n} = 4 - t, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_nF_{n+1}} = t - 1, \prod_{n=2}^{\infty}(1 + \frac{(-1)^n}{F_n^2}) = t$.

2. *Show by induction that* $f(n) = \sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n$.

3. *Prove that for any natural* N, $\sqrt{m\sqrt{(m+1)\sqrt{\ldots\sqrt{N}}}} < m + 1$.

4. *We build the exponential tower* $X = \sqrt{2}^X$ *by defining* $a_0 = 1$ *and* $a_{n+1} = \sqrt{2}^{a_n}, n \geq 0$. *Show that the sequence* a_n *is increasing monotonically and bounded above by* 2.

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5. *n circles are given in the plane. They divide the plane into parts. Show that you can close the plane with two colours, so that no parts with a common boundary line are coloured the same way. Such a colouring is called a proper colouring.*

6. *Find a closed from for the expression with n radicals defined as*

$$a_n = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}.$$

7. *Let $\alpha \in \mathbb{R}$ such that $\alpha + \frac{1}{\alpha} \in \mathbb{Z}$. Show that $\alpha^n + \frac{1}{\alpha^n} \in \mathbb{Z}$, for any $n \in \mathbb{N}$.*

8. *Prove that*

$$1 < \frac{1}{n+1} + \cdots + \frac{1}{3n+1} < 2.$$

9. *Prove that*

$$(n+1)(n+2)\cdots 2n = 2^n.1.3.5.\cdots.(2n-1),$$

for all $n \in \mathbb{N}$.

10. *Prove that if $z + \frac{1}{z} = 2\cos\alpha$ then, $z^n + \frac{1}{z^n} = 2\cos n\alpha, \forall n \in \mathbb{N}$.*

11. *Consider all possible subsets of the set $\{1, 2, \ldots, N\}$, which do not contain any neighbouring points P. Prove that the sum of the squares of the products of all numbers in these subsets is $(N+1)! - 1$.*

12. *Let $a_1, a_2, \ldots, a_n$ be positive integers such that $a_1 \leq a_2 \ldots \leq a_n$. Prove that if $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1$, then $a_n = 2^{n!}$.*

13. *Prove that $3^{n+1} \mid 2^{3^n} + 1, \forall n \geq 0$.*

14. *Let $n = 2^k$, prove that we can select an integer from any $(2n-1)$ integers such that their sum is divisble by n.*

15. *Prove that all numbers of the form $1007, 10017, 10117, \ldots$ are divisible by 53.*

16. *Prove that all numbers of the form $12008, 120308, 1203308, \ldots$ are divisible by 19.*

17. *Let x_1, X_2 be the roots of the equation $x^2 + px - 1 = 0$, p odd and set $y_n = x_1^n + x_2^n, n \geq 0$, then prove that y_n and y_{n+1} are coprime integers.*

18. *Prove that the cube of any integer can be written as the difference of two squares.*