INEQUALITIES PROBLEMS

MANJIL P. SAIKIA

NEHU Pre-INMO Camp, 2013

1. Some Standard Inequalities

Theorem 1.1. (**Schur**) *Let x, y, z be nonnegative real numbers. For any $r > 0$, we have*

$$\sum_{\text{cyclic}} x^r(x-y)(x-z) \geq 0.$$

Theorem 1.2. (**Muirhead**) *Let $a_1, a_2, a_3, b_1, b_2, b_3$ be real numbers such that*

$$a_1 \geq a_2 \geq a_3 \geq 0, b_1 \geq b_2 \geq b_3 \geq 0, a_1 \geq b_1, a_1+a_2 \geq b_1+b_2, a_1+a_2+a_3 = b_1+b_2+b_3.$$

Let x, y, z be positive real numbers. Then, we have $\sum_{\text{sym}} x^{a_1}y^{a_2}z^{a_3} \geq \sum_{\text{sym}} x^{b_1}y^{b_2}z^{b_3}$.

Theorem 1.3. (**The Cauchy-Schwarz inequality**) *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers. Then,*

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1b_1 + \cdots + a_nb_n)^2.$$

Theorem 1.4. (**AM-GM inequality**) *Let $a_1, \cdots, a_n$ be positive real numbers. Then, we have*

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Theorem 1.5. (**Weighted AM-GM inequality**) *Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n > 0$, we have*

$$\omega_1\, x_1 + \cdots + \omega_n\, x_n \geq {x_1}^{\omega_1} \cdots {x_n}^{\omega_n}.$$

Theorem 1.6. (**Hölder's inequality**) *Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have*

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} {x_{ij}}^{\omega_j} \right).$$

Theorem 1.7. (**Power Mean inequality**) *Let $x_1, \cdots, x_n > 0$. The power mean of order r is defined by*

$$M_{(x_1,\cdots,x_n)}(0) = \sqrt[n]{x_1 \cdots x_n}\,, \quad M_{(x_1,\cdots,x_n)}(r) = \left(\frac{x_1^r + \cdots + {x_n}^r}{n}\right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then, $M_{(x_1,\cdots,x_n)} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and monotone increasing.

Theorem 1.8. (**Majorization inequality**) *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a convex function. Suppose that $(x_1, \cdots, x_n)$ majorizes $(y_1, \cdots, y_n)$, where $x_1, \cdots, x_n, y_1, \cdots, y_n \in [a,b]$. Then, we obtain*

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

Theorem 1.9. (**Bernoulli's inequality**) *For all $r \geq 1$ and $x \geq -1$, we have*

$$(1+x)^r \geq 1 + rx.$$

Definition 1.1. (**Symmetric Means**) *For given arbitrary real numbers $x_1, \cdots, x_n$, the coefficient of t^{n-i} in the polynomial $(t+x_1)\cdots(t+x_n)$ is called the i-th elementary symmetric function σ_i. This means that*

$$(t+x_1)\cdots(t+x_n) = \sigma_0 t^n + \sigma_1 t^{n-1} + \cdots + \sigma_{n-1} t + \sigma_n.$$

For $i \in \{0, 1, \cdots, n\}$, the i-th elementary symmetric mean S_i is defined by

$$S_i = \frac{\sigma_i}{\binom{n}{i}}.$$

Theorem 1.10. *Let $x_1, \ldots, x_n > 0$. For $i \in \{1, \cdots, n\}$, we have*

(1) (**Newton's inequality**) $\frac{S_i}{S_{i+1}} \geq \frac{S_{i-1}}{S_i}$,

(2) (**Maclaurin's inequality**) ${S_i}^{\frac{1}{i}} \geq {S_{i+1}}^{\frac{1}{i+1}}$.

Theorem 1.11. (**Rearrangement inequality**) *Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. For any permutation σ of $\{1, \ldots, n\}$, we have*

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i y_{\sigma(i)} \geq \sum_{i=1}^{n} x_i y_{n+1-i}.$$

Theorem 1.12. (**Chebyshev's inequality**) *Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. We have*

$$\frac{x_1 y_1 + \cdots + x_n y_n}{n} \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Theorem 1.13. (**Hölder's inequality**) *Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be positive real numbers. Suppose that $p > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} {y_i}^q\right)^{\frac{1}{q}}$$

Theorem 1.14. (**Minkowski's inequality**) *If $x_1, \cdots, x_n, y_1, \cdots, y_n > 0$ and $p > 1$, then*

$$\left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} {y_i}^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{\frac{1}{p}}$$

2. Some Problems

Problem 1. *Let a, b, c be the lengths of the sides of a triangle. Suppose that $u = a^2 + b^2 + c^2$ and $v = (a + b + c)^2$. Prove that*

$$\frac{1}{3} \leq \frac{u}{v} < \frac{1}{2}$$

and that the fraction 1/2 on the right cannot be replaced by a smaller number.

Problem 2. *Let a and b be positive real numbers. Prove that the inequality*

$$\frac{(a+b)^3}{a^2 b} \geq \frac{27}{4}$$

holds. When does equality hold?

Problem 3. *Suppose x and y are positive real numbers such that x + 2y = 1. Prove that*

$$\frac{1}{x} + \frac{2}{y} \geq \frac{25}{1 + 48xy^2}.$$

Problem 4. *Let $a_1, a_2, \ldots, a_n$ be distinct positive integers. Prove that*

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Problem 5. *Let x,y,z be positive real numbers and $xyz \geq 1$. Prove that*

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0$$

Problem 6. *Prove that, for every positive integer n:*

$$\frac{1}{11} + \frac{2}{21} + \frac{3}{31} + \cdots \frac{n}{10n+1} < \frac{n}{10}$$

Problem 7. *Let a, b, u, v be nonnegative numbers. Suppose that $a^5 + b^5 \leq 1$ and $u^5 + v^5 \leq 1$. Prove that*

$$a^2 u^3 + b^2 v^3 \leq 1.$$

Problem 8. *Let n be a positive integer and x ¿ 0. Prove that*

$$(1+x)^{n+1} \geq \frac{(n+1)^(n+1)}{n^n} x.$$

Problem 9. *Prove that*
$$(\sqrt{n-1}+\sqrt{n}+\sqrt{n+1})^2<9n.$$

Problem 10. *Suppose that* $a_1<a_2<\cdots<a_n$. *Prove that*
$$a_1a_2^4+a_2a_3^4+\cdots+a_na_1^4\geq a_2a_1^4+a_3a_2^4+\cdots+a_1a_n^4.$$

Problem 11. *Let* a,b,c *be positive real numbers with* $ab+bc+ca=abc$. *Prove that*
$$\frac{a^4+b^4}{ab(a^3+b^3)}+\frac{b^4+c^4}{bc(b^3+c^3)}+\frac{c^4+a^4}{ca(c^3+a^3)}\geq 1.$$

Problem 12. *Let* a,b,c *be the lengths of the sides of a triangle. Prove that*
$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b}\leq\sqrt{a}+\sqrt{b}+\sqrt{c}.$$

Problem 13. *Let* a,b,c *be the lengths of the sides of a triangle. Prove the inequality*
$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}\leq 3.$$

Problem 14. *Let* ABC *be a triangle. Prove that*
$$\sin 3A+\sin 3B+\sin 3C\leq\frac{3\sqrt{3}}{2}.$$

Problem 15. (Chebyshev's Inequality) Let $x_1,\cdots,x_n$ and $y_1,\cdots y_n$ be two monotone increasing sequences of real numbers:
$$x_1\leq\cdots\leq x_n,\ y_1\leq\cdots\leq y_n.$$
Then, we have the estimation
$$\sum_{i=1}^{n}x_iy_i\geq\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i\right).$$

Problem 16. *Let* $a_1,\cdots,a_n,b_1,\cdots,b_n$ *be positive real numbers such that* $a_1+\cdots+a_n=b_1+\cdots+b_n$. *Show that*
$$\frac{{a_1}^2}{a_1+b_1}+\cdots+\frac{{a_n}^2}{a_n+b_n}\geq\frac{a_1+\cdots+a_n}{2}.$$

Problem 17. *Let* $a,b,c,d\geq 0$ *with* $ab+bc+cd+da=1$. *show that*
$$\frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c}\geq\frac{1}{3}.$$

Problem 18. (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that
$$a^2+b^2+c^2\geq 4\sqrt{3}S.$$

Problem 19. (Hadwiger-Finsler Inequality) For any triangle ABC with sides a, b, c and area F, the following inequality holds.
$$2ab+2bc+2ca-(a^2+b^2+c^2)\geq 4\sqrt{3}F.$$

Problem 20. (Tsintsifas) Let p,q,r be positive real numbers and let a,b,c denote the sides of a triangle with area F. Then, we have
$$\frac{p}{q+r}a^2+\frac{q}{r+p}b^2+\frac{r}{p+q}c^2\geq 2\sqrt{3}F.$$

Problem 21. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {b_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) + {c_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) \geq 16F_1F_2.$$

Problem 22. *Let $x_1, \cdots, x_n$ be arbitrary real numbers. Prove the inequality.*

$$\frac{x_1}{1+{x_1}^2} + \frac{x_2}{1+{x_1}^2+{x_2}^2} + \cdots + \frac{x_n}{1+{x_1}^2+\cdots+{x_n}^2} < \sqrt{n}.$$

Problem 23. *Let x, y, and z be positive numbers such that $xyz \geq 1$. Prove that*

$$\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{y^5+z^2+x^2} + \frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$

Problem 24. *Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,*

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Problem 25. *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

Problem 26. (Muirhead's Theorem) Let $a_1, a_2, a_3, b_1, b_2, b_3$ be real numbers such that

$$a_1 \geq a_2 \geq a_3 \geq 0, b_1 \geq b_2 \geq b_3 \geq 0, a_1 \geq b_1, a_1+a_2 \geq b_1+b_2, a_1+a_2+a_3 = b_1+b_2+b_3.$$

Let x, y, z be positive real numbers. Then, we have

$$\sum_{\text{sym}} x^{a_1}y^{a_2}z^{a_3} \geq \sum_{\text{sym}} x^{b_1}y^{b_2}z^{b_3}.$$

Problem 27. *If m_a,m_b,m_c are medians and r_a,r_b,r_c the exradii of a triangle, prove that*

$$\frac{r_ar_b}{m_am_b} + \frac{r_br_c}{m_bm_c} + \frac{r_cr_a}{m_cm_a} \geq 3.$$

Problem 28. *Prove that, for all $a, b, c > 0$,*

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

Problem 29. (Hölder's Inequality) Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have

$$\prod_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij}\right)^{\omega_j} \geq \sum_{i=1}^{m}\left(\prod_{j=1}^{n} {x_{ij}}^{\omega_j}\right).$$

Problem 30. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a* **continuous** *function. Then, the followings are equivalent.*

(1) For all $n \in \mathbb{N}$, the following inequality holds.

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a, b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1+\cdots+\omega_n = 1$.

(2) For all $n \in \mathbb{N}$, the following inequality holds.

$$r_1 f(x_1) + \cdots + r_n f(x_n) \geq f(r_1\, x_1 + \cdots + r_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a,b]$ *and* $r_1, \cdots, r_n \in \mathbb{Q}^+$ *with* $r_1 + \cdots + r_n = 1$.
(3) For all $N \in \mathbb{N}$, *the following inequality holds.*

$$\frac{f(y_1) + \cdots + f(y_N)}{N} \geq f\left(\frac{y_1 + \cdots + y_N}{N}\right)$$

for all $y_1, \cdots, y_N \in [a,b]$.
(4) For all $k \in \{0, 1, 2, \cdots\}$, *the following inequality holds.*

$$\frac{f(y_1) + \cdots + f(y_{2^k})}{2^k} \geq f\left(\frac{y_1 + \cdots + y_{2^k}}{2^k}\right)$$

for all $y_1, \cdots, y_{2^k} \in [a,b]$.
(5) We have $\frac{1}{2}f(x) + \frac{1}{2}f(y) \geq f\left(\frac{x+y}{2}\right)$ *for all* $x, y \in [a,b]$.
(6) We have $\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$ *for all* $x, y \in [a,b]$ *and* $\lambda \in (0,1)$.

***

Department of Mathematical Sciences, Tezpur University, Napaam-784028, Sonitpur, Assam, India
E-mail address: manjil@gonitsora.com