Number Theory Problems

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1. Given seven distinct integers that add up to 100, prove that some three of them add upto at least 50.

2. Consider $2n$ distinct positive integers $a_1, a_2, \ldots, a_{2n}$ not exceeding n^2, where $n > 2$. Prove that some three of the differences $a_i - a_j$ are equal.

3. Consider seven distinct positive integers not exceeding 1706. Prove that there are three of them say a, b, c such that $a < b + c < 4a$.

4. Prove that for every positive integer m, there is a positive integer n such that $m + n + 1$ is a perfect square and $mn + 1$ is a perfect cube.

5. Prove that $x^2 + y^2 + 1 = z^2$ has infinitely many integer solutions.

6. Prove that $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$ is divisible by p for all natural numbers n iff p is a prime.

7. Prove that a repuit greater than 1 cannot be a square of an integer.

8. Prove that for any natural number n we have
$$\lfloor \frac{n+2^0}{2^1} \rfloor + \lfloor \frac{n+2^1}{2^2} \rfloor + \lfloor \frac{n+2^2}{2^3} \rfloor + \ldots = n.$$

9. Let $(p, q) = 1$ then find
$$\{\frac{p}{q}\} + \{\frac{2p}{q}\} + \ldots + \{\frac{(q-1)p}{q}\}.$$

10. Let $(p, q) = 1$ then show that
$$\lfloor \frac{p}{q} \rfloor + \lfloor \frac{2p}{q} \rfloor + \ldots + \lfloor \frac{(q-1)p}{q} \rfloor = \frac{(p-1)(q-1)}{2}.$$

11. Find all pairs of possible positive integers that satisfy $x^2 - y! = 2001$.

12. Let n be an integer greater than 1. Then prove that

    a. $1! + 2! + 3! + \ldots + n!$ is a perfect power iff $n = 3$.
    b. $(1!)^3 + (2!)^3 + (3!)^3 + \ldots + (n!)^3$ is a perfect power iff $n = 3$.

13. Let A be the sum of the digits of the number 4444^{4444}, and B be the sum of the digits of A. Compute the sum of the digits of B.

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14. Show that $\frac{(2m)!(2n)!}{(m+n)!m!n!}$ is an integer.

15. Prove that $36^{41} + 41^{36}$ is divisible by 77.

16. Let $p > 2$ be a prime number and let $a_k \in \{0, 1, \cdots, p^2 - 1\}$ denote the value of k^p modulo p^2. Prove that $\sum_{k=1}^{p-1} a_k = \frac{p^3-p^2}{2}$.

17. If k is a positive integer, then show that $\lfloor \sqrt{k^2+1} + \ldots + \sqrt{k^2+2k} \rfloor = 2k^2 + 2k$.

18. Every natural number $n > 6$ can be written as a sum of distinct primes.

19. Prove that every integer can be written as a sum of five distinct cubes.

20. Consider an arithmetic progression with ratio between 1 and 2000. Show that the progression does not contain more than 10 consecutive primes.

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