On some simple Geometrical applications

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"Inspiration is needed in geometry, just as much as in poetry."
-- Aleksandr Pushkin

Abstract: This article is based on a lecture given at Kaliabor College Kaliabor(India) for the mathematical Olympiad enthusiasts.This article discusses some simple geometrical properties with some applications.

1. Warm Up Problem: On the sides BC, CA, AB of triangle points D, *E* and *F* are taken in such a way so that $\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}=2$. Show that the area of the triangle formed by the lines AD, BE, CF is one seventh the area of the triangle ABC.

2. Some important results:

2.1(Ceva) If the points D, E, F are taken on the sides BC, CA, AB of triangle ABC such that the lines *AD,BE,CF* are concurrent at a point *O*, then $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB}=1$.

2.2 (Menelaus) If a transversal cuts the sides BC,CA,AB (suitably extended) of triangle *ABC* in points *D,E,F* respectively then $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB}=-1$

2.3 (Stewart) If point *D* divides the base *BC* of triangle *ABC* in the ratio $\frac{BD}{DC}=\frac{m}{n}$ then we have $mAB^2+nAC^2=mBD^2+nCD^2+(m+n)AD^2$.

2.4 (Ptolemy) The rectangle obtained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by the pairs of its opposite sides.

2.5 (Euler) In any triangle *ABC* , $OI^2=R^2-2Rr$ where *O, I* are the centres and *R, r* are the radii , respectively of the circumcircle and incircle of triangle ABC.

2.6 (Apollonius) If *AD* is a median of the triangle *ABC*, where *D* is a point on *BC* then $AB^2+AC^2=2(BD^2+AD^2)$

3. Applications: Solve the following problems by applying the theorems mentioned above or otherwise.

3.1 Prove that the altitudes of a triangle are concurrent.

3.2 Points *E,F* on the sides *CA,AB* of triangle *ABC* such that *FE* is parallel to *BC; BE, CF* intersect at *X*. Prove that *AX* is a median of triangle *ABC*.

3.3 The external bisector of angle *A* of triangle *ABC* meets *BC* produced at *L* and the internal bisector of angle *B* meets *CA* at *M*. If *LM* meets *AB* at *R* then prove that *CR* bisects the angle *C*.

3.4 (Cosine rule) In triangle *ABC* , *AD* is perpendicular to *BC*, prove that $AB^2 = BC^2 + CA^2 + 2BC.DC$ when angle *C* is obtuse.

3.5 Consider an isosceles triangle. Let *r* be the radius of its circumcircle and *p* the radius of the inscribed circle. Prove that the distance between the centres of these circles is $d = [r(r-2p)]^{0.5}$.

3.6 (Brahmagupta) If in triangle *ABC* , *AD* is the altitude and *AE* is the diameter of the circumcircle through *A*, then prove that $AB.AC = AD.AE$.

3.7 In triangle *ABC*, prove that $m_a + m_b + m_c < a+b+c < \frac{4}{3}(m_a + m_b + m_c)$, where *a,b,c* are the sides of the triangle and m_a, m_b, m_c are the medians of the triangle.

3.8 In triangle *ABC* , *G* is the centroid , then prove that $AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$.

3.9 The points A_0, A_1, A_2, A_3, A_4 divide a unit circle into five equal parts . Prove that the chords A_0A_1, A_0A_2 satisfy $(A_0A_1.A_0A_2)^2 = 5$.

3.10 Prove *2.6.*

4. Geometrical Inequalities:

4.1 (Andreescu-Enescu Lemma/Engel's Lemma) If $a_i(i = 1, 2, ...n)$ and $b_i(i = 1, 2, ..., n)$ are real numbers with $b_i(i = 1, 2, ..., n)$ are not equal not *0*, then we have , $\sum \frac{a_i^2}{b_i} \geq \frac{(\sum a_i)^2}{\sum b_i}, (i = 1, 2, 3, ..., n)$ where equality occurs when $a_i = kb_i, k \in Z(i = 1, 2, ..., n)$.

4.2 (Nesbitt) If *a,b,c* are the sides of a triangle , show that $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$ where equality occurs in case of an equilateral triangle.

4.3 (IMO 1983/6) Let *a,b,c* be the lengths of the sides of a triangle then prove that $a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$.

4.4 Let the bisector of angle *C* of triangle *ABC* meet the side *AB* at *D*. Prove that $CD^2 < AC.BC$.

4.5 (IMO 1995/2) Let *a,b,c* be positive reals such that *abc=1*, then prove that $\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$.

4.6 Give a geometrical significance of the **A.M-G.M** inequality for two variables.

5. Parting Gift:

(Erdos-Mordell) If from appoint *P* inside a triangle *ABC*, perpendiculars PH_1, PH_2, PH_3 are drawn to its sides then $PA + PB + PC \geq 2(PH_1 + PH_2 + PH_3)$.

[P.S.: Please note that the solutions of the problems are not given in this version of the lecture!]