

# **THE PURSUIT OF JOY-I**

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### **Dedication**

The authors wish to dedicate this book to their parents and teachers without whose support this would not have been possible.

### **Acknowledgements**

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## Preface

Mathematical Olympiad activities have been going on all over the world for more than a 100 years now. In India too it started more than 25 years ago, and in Assam, the Assam Academy of Mathematics has been organizing state level mathematical olympiads for more than two decades now. Olympiad problems have a beauty and originality that is often not encountered in a textbook stereotype problem. To bridge the gap between this school level mathematics and Olympiad mathematics, there are numerous books and materials available both print and online. However, the sad part is that very often this material is so scattered that the students of this part of the country have a very hard time collecting them.

To bridge this gap, there are numerous publications by the Assam Academy of Mathematics which focuses on specific parts mathematics. However, there has been a long felt need for a good problem book containing just a few of the Olympiad gems from which both Olympiad enthusiasts and mathematics lovers can pluck out a problem and relish in its solution. In order to fulfill that need we have started this series of books called *The Pursuit of Joy*. The current volume is the first in a planned five volume set. This volume concerns mainly with problems and solutions of a few problems from Number Theory and Inequalities.

The approach that we follow in this book is that of a purely problems oriented book. We begin each section with a few results without proofs, and then we list out a few problems. The problems are varied in their difficulty level, and are mostly taken from various National Olympiads held all the world. Infact, a few IMO (International Mathematical Olympiad) problems are also given from time to time. Each problem section is followed by solutions of several of the problems, and hints to the others that are not solved. However, the reader is asked not to look directly into the solution without giving the problem a shot. It should be kept in mind that the solution we give here may not be the only solution to that particular problem. The readers are asked to try to look for other solutions that they may think of.

The Inequalities part of the book reflect the love for Geometric Inequalities by the second author and contains many gems of Geometric Inequalities, almost each of them with a solution. We end the book with a list of more than a hundred unsolved problems involving inequalities. These problems are again collected from a variety of sources, and are varied in their difficulty level. Do not get disheartened if you are not able to solve these, some of them are really difficult. The secret to any type of problem solving is perseverance and a tenacity to remember other solved problems. Olympiad mathematics is an enjoyable hobby,

and hopefully the readers will enjoy the collection of problems that we present here.

It is impossible on our part to mention all the problem proposers whose problems we have included in our book, but we are deeply grateful to each and every one of them. Very few of the problems that are included in the book are original problems proposed by us. The interested readers may like to look into some of the References that we mention at the end. We have also set up a website for the book where we will provide some other materials from time to time, The URL of the website is <http://gonitsora.com/joy/> . We would also be grateful to the readers if they point out any errors in this book. We would also like to hear about any suggestions for the further improvement of the later volumes in this series. We can be contacted at [joy@gonitsora.com](mailto:joy@gonitsora.com)

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"Mighty are numbers, joined with art resistless." - Euprides

## 1. HEURISTICS OF PROBLEM SOLVING

Strategy or tactics in problem solving is called *heuristics*. Here is a summary taken from *Problem-Solving Through Problems* by Loren C. Larson.

1. Search for a pattern.
2. Draw a figure.
3. Formulate an equivalent problem.
4. Modify the problem.
5. Choose effective notation.
6. Exploit symmetry.
7. Divide into cases.
8. Work backwards.
9. Argue by contradiction.
10. Check for parity.
11. Consider extreme cases.
12. Generalize.

## 2. NUMBER THEORY

Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is.

- P. Erdős

## 2.1. Preliminaries.

**Property 2.1.** *If  $b = aq$  for some  $q \in \mathbb{Z}$ , then  $a$  divides  $b$ , and we write  $a \mid b$ .*

**Property 2.2.** *(Fundamental Properties of the Divisibility Relation)*

- $a \mid b, b \mid c \Rightarrow a \mid c$ .
- $d \mid a, d \mid b \Rightarrow d \mid ax + by$ .

**Property 2.3.** *(Division Algorithm) Every integer  $a$  is uniquely representable by the positive integer  $b$  in the form  $a = bq + r, 0 \leq r < b$ .*

**Property 2.4.** *(Euclidean Algorithm) In the above representation of integers  $\gcd(a, b) = \gcd(b, r)$ .*

**Theorem 2.1.** *(Bézout's Identity) The  $\gcd(a, b)$  can be represented by a linear combination of  $a$  and  $b$  with integral coefficients such that, there are  $x, y \in \mathbb{Z}$ , so that  $\gcd(a, b) = ax + by$ .*

**Theorem 2.2.** *(Euclid's Lemma) If  $p$  is a prime,  $p \mid ab \Rightarrow p \mid a$  or  $p \mid b$ .*

**Theorem 2.3.** *(Fundamental Theorem of Arithmetic) Every positive integer can be uniquely represented as a product of primes.*

**Theorem 2.4.** *(Euclid) There are infinitely many primes.*

**Property 2.5.**  $n! + 2, n! + 3, n! + 4, \dots, n! + n$  are  $(n - 1)$  consecutive composite integers.

**Property 2.6.** *The smallest prime factor of a nonprime  $n$  is  $\leq \sqrt{n}$ .*

**Theorem 2.5.** *All pairwise prime triples of integers satisfying  $x^2 + y^2 = z^2$  are given by  $x = |u^2 - v^2|$ ,  $y = 2uv$  and  $z = u^2 + v^2$ ,  $\gcd(u, v) = 1$  and  $u - v$  is not divisible by 2.*

**Notation 2.1.** *(Congruences) If  $m \mid a - b$  then we write  $a \equiv b \pmod{m}$ .*



Congruences can be added, subtracted and multiplied in a usual manner but they cannot be divided always.

**Theorem 2.6. (Fermat's Little Theorem)** *Let  $a$  be a positive integer and  $p$  be a prime, then  $a^p \equiv a \pmod{p}$ .*

The converse is however not valid.

**Theorem 2.7. (Wilson's Theorem)** *If  $p$  is a prime, then  $(p-1) \equiv -1 \pmod{p}$ .*

**Definition 2.1. (Euler's totient function)**  $\phi(m)$  denotes the number of numbers less than  $m$  which are prime to  $m$ .

**Property 2.7.**  $\gcd(a, m) = 1 \Rightarrow a^{\phi(m)} \equiv 1 \pmod{m}$ .

**Theorem 2.8. (Euler)** *If  $a$  and  $m$  be relatively prime positive integers then,  $a^{\phi(m)} \equiv 1 \pmod{m}$ .*

**Property 2.8. (Sophie Germain Identity)**  $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$ .

**Definition 2.2.** We define  $\text{ord}_p(n)$ , by the nonnegative integer  $k$  such that  $p^k \parallel n$ . Then,

$$n = \prod_{p:\text{prime}} p^{\text{ord}_p(n)}$$

**Theorem 2.9.** *Let  $A$  and  $B$  be positive integers, then  $A$  is a multiple of  $B$  iff  $\text{ord}_p(A) \geq \text{ord}_p(B)$  holds for all primes  $p$ .*

**Notation 2.2.**  $\lfloor x \rfloor$  is the greatest integer less than or equal to  $x$ .  $\lfloor x \rfloor$  is read as floor of  $x$ .

**Theorem 2.10. (De Polignac)**  $\text{ord}_p(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor$ .

**Property 2.9.** •  $\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$ .

•  $\lfloor \frac{\lfloor x \rfloor}{n} \rfloor = \lfloor \frac{x}{n} \rfloor$ .

•  $\lfloor x + \frac{1}{2} \rfloor =$  the integer nearest to  $x$ .

**Theorem 2.11. (Hermite)**  $\lfloor nx \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{n} \rfloor + \dots + \lfloor x + \frac{n-1}{n} \rfloor$ .

**Notation 2.3.**  $\tau(n)$  denotes the number of divisors of the nonnegative integer  $n$ .

**Notation 2.4.**  $\sigma(n)$  denotes the sum of the divisors of the nonnegative integer  $n$ .

**Theorem 2.12.** *If  $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$  is a prime decomposition of  $n$ , then,*

$$\tau(n) = (a_1 + 1)(a_2 + 1) \dots (a_k + 1)$$

**Theorem 2.13.** *With the notation of the previous theorem we have,  $(2a_1 + 1)(2a_2 + 1) \cdots (2a_k + 1)$  distinct pairs of ordered positive integers  $(a, b)$  with  $\text{lcm}(a, b) = n$ .*

**Theorem 2.14.** *For any positive integer  $n$ ,  $\prod_{d|n} d = n^{\frac{\tau(n)}{2}}$ .*

**Theorem 2.15.** *With the same notation as the above three theorems we have,*

$$\sigma(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdots \frac{p_k^{a_k+1} - 1}{p_k - 1}.$$

## 2.2. Problems.

- Problem 1.**
- Find all natural numbers  $n$  for which 7 divides  $2^n - 1$ .
  - Prove that there is no natural number  $n$  for which 7 divides  $2^n + 1$ .

**Problem 2.** Prove that for each positive integer  $n$ , there are pairwise relatively prime integers  $k_0, k_1, \dots, k_n$ , all strictly greater than 1, such that  $k_0 k_1 \cdots k_n - 1$  is the product of two consecutive integers.

**Problem 3.** Determine the values of the positive integer  $n$  for which

$$\sqrt{\frac{9n-1}{n+7}}$$

is rational.

**Problem 4.** Suppose  $a, b$  are integers satisfying  $24a^2 + 1 = b^2$ . Prove that exactly one of  $a, b$  is divisible by 5.

**Problem 5.** Let  $x = abcd$  be a 4-digit number such that the last 4 digits of  $x^2$  are also  $abcd$ . Find all possible values of  $x$ .

**Problem 6.** Let  $a$  and  $b$  be positive integers and let  $u = a + b$  and  $v = \text{lcm}(a, b)$ . Prove that

$$\gcd(u, v) = \gcd(a, b).$$

**Problem 7.** Determine the units digit of the numbers  $a^2, b^2$  and  $ab$  (in base 10), where

$$a = 2^{2002} + 3^{2002} + 4^{2002} + 5^{2002}$$

and

$$b = 3^1 + 3^2 + 3^3 + \cdots + 3^{2002}.$$

**Problem 8.** Let  $p$  be an odd prime. Let  $k$  be a positive integer such that  $\sqrt{k^2 - pk}$  also a positive integer. Find  $k$ .

**Problem 9.** An integer  $n > 1$  has the property, that for every (positive) divisor  $d$  of  $n$ ,  $d + 1$  is a divisor of  $n + 1$ . Prove that  $n$  is prime.

**Problem 10.** Let  $N$  be the number of ordered pairs  $(x, y)$  of integers such that

$$x^2 + xy + y^2 \leq 2007.$$

Prove that  $N$  is odd.

**Problem 11.** Given a finite set  $P$  of prime numbers, there exists a positive integer  $x$  such that it can be written in the form  $a^p + b^p$  ( $a, b$  are positive integers), for each  $p \in P$ , and cannot be written in that form for each  $p$  not in  $P$ .

**Problem 12.** (Primitive Pythagoras Triangles) Let  $x, y, z \in \mathbb{N}$  with  $x^2 + y^2 = z^2$ ,  $\gcd(x, y) = 1$ , and  $x \equiv 0 \pmod{2}$ . Then, there exists positive integers  $p$  and  $q$  such that  $\gcd(p, q) = 1$  and

$$(x, y, z) = (2pq, p^2 - q^2, p^2 + q^2).$$

**Problem 13.** The equation  $x^4 + y^4 = z^2$  has no solution in positive integers.

**Problem 14.** Let  $a$  and  $b$  be positive integers. Show that if  $4ab - 1$  divides  $(4a^2 - 1)^2$ , then  $a = b$ .

**Problem 15.** Let  $\mathbb{N} = \{1, 2, 3, \dots\}$  denote the set of positive integers. Find all functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $m, n \in \mathbb{N}$ :  $f(2) = 2$ ,  $f(mn) = f(m)f(n)$ ,  $f(n+1) > f(n)$ .

**Problem 16.** A natural number  $p > 1$  is a prime if and only if  $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$  is divisible by  $p$  for every non-negative  $n$ , where  $\binom{n}{p}$  is the number of different ways in which we can choose  $p$  out of  $n$  elements and  $\lfloor x \rfloor$  is the greatest integer not exceeding the real number  $x$ .

**Problem 17.** Prove that the sequence  $a_n = \binom{m}{x} \pmod{m}$  is periodic, where  $x, m \in \mathbb{N}$ .

**Problem 18.** Prove that for a natural number  $m = \prod_{i=1}^k p_i^{b_i}$ , the sequence  $a_n = \binom{n}{m} \pmod{m}$  has a period of minimal length,

$$l(m) = \prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor + b_i}$$

**Problem 19.** Let  $n$  be relatively prime to  $m$ . Then, show that

$$\binom{n}{m} \equiv \binom{n-1}{m} \pmod{m}.$$

**Problem 20.** Let  $m$  be even. Then show that for every integer  $k$  we have,

$$\binom{m+k}{m} \equiv \binom{l(m) - 1 - k}{m} \pmod{m}.$$

**Problem 21.** Let  $d(n)$  denote the number of positive divisors of the number  $n$ . Prove that the sequence  $d(n^2 + 1)$  does not become strictly monotonic from some point onwards.

**Problem 22.** Prove that  $d((n^2 + 1)^2)$  does not become monotonic from any given point onwards.

**Problem 23.** Suppose that  $a$  and  $b$  are distinct real numbers such that:

$$(1) \quad a - b, a^2 - b^2, \dots, a^k - b^k, \dots$$

are all integers. Show that  $a$  and  $b$  are integers.

**Problem 24.** Prove that there are no integers  $x$  and  $y$  satisfying  $x^2 = y^5 - 4$ .

**Problem 25.** Suppose the set  $M = \{1, 2, \dots, n\}$  is partitioned into  $t$  disjoint subsets  $M_1, \dots, M_t$ . Show that if  $n \geq \lfloor t! \cdot e \rfloor$  then at least one class  $M_z$  contains three elements  $x_i, x_j, x_k$  with the property that  $x_i - x_j = x_k$ .

**Problem 26.** Let  $p$  be a prime number of the form  $4k + 1$ . Show that

$$\sum_{i=1}^{p-1} \left( \left\lfloor \frac{2i^2}{p} \right\rfloor - 2 \left\lfloor \frac{i^2}{p} \right\rfloor \right) = \frac{p-1}{2}.$$

**Problem 27.** Let  $a$  and  $b$  be positive integers such that  $ab + 1$  divides  $a^2 + b^2$ . Show that

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.

**Problem 28.** Suppose that  $p$  is an odd prime. Prove that

$$\sum_{j=0}^p \binom{p}{j} \binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

**Problem 29.** Let  $n$  be a prime and  $a_1 < a_2 < \dots < a_n$  be integers. Prove that  $a_1, a_2, \dots, a_n$  is an arithmetic progression if and only if there exists a partition of  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$  into  $n$  sets  $A_1, A_2, \dots, A_n$  so that

$$a_1 + A_1 = a_2 + A_2 = \dots = a_n + A_n,$$

where  $x + A = \{x + a \mid a \in A\}$ .

**Problem 30.** Consider the set of all five-digit numbers whose decimal representation is a permutation of the digits 1, 2, 3, 4, 5. Prove that this set can be divided into two groups, in such a way that the sum of the squares of the numbers in each group is the same.

### 2.3. Solutions.

**Solution 1.** • Since  $2^3 \equiv 8 \equiv 1 \pmod{7}$ , this means  $2^3 \pmod{7}$  is periodic with period 3. It suffices to consider following three cases:

1) If  $n = 3k$ , then  $2^n - 1 \equiv 2^{3k} - 1 \equiv (2^3)^k - 1 \equiv 1^k - 1 \equiv 1 - 1 \equiv 0 \pmod{7}$ .

2) If  $n = 3k + 1$ , then  $2^n - 1 \equiv 2^{3k+1} - 1 \equiv 2 \times 2^{3k} - 1 \equiv 2 - 1 \equiv 1 \pmod{7}$ .

3) If  $n = 3k + 2$ , then  $2^n - 1 \equiv 4 \times 2^{3k} - 1 \equiv 4 - 1 \equiv 3 \pmod{7}$ . Therefore, we conclude that  $2^n - 1$  is divisible by 7 if and only if  $n=3k$ .

• The proof of this is similar to the above.

**Solution 2.** This problem can be solved by induction.

For  $n=1$ , we may take  $k_0 = 3$  and  $k_1 = 7$ .

Let us assume now that for a certain  $n$  there are pairwise relatively prime integers  $1 < k_0 < k_1 < \dots < k_n$  such that

$k_0 k_1 \dots k_n - 1 = a_n(a_n - 1)$ , for some positive integer  $a_n$ .

Now  $k_0 k_1 \dots k_n = a_n^2 - a_n + 1$ . Then choosing  $k_{n+1} = a_n^2 + a_n + 1$  yields

$$k_0 k_1 \dots k_{n+1} = (a_n^2 - a_n + 1)(a_n^2 + a_n + 1) = a_n^4 + a_n^2 + 1.$$

Thus

$$k_0 k_1 \dots k_{n+1} - 1 = a_n^2(a_n^2 + 1).$$

So,  $k_0 k_1 \dots k_{n+1} - 1$  is the product of the two consecutive integers  $a_n^2$  and  $a_n^2 + 1$ . Moreover,

$$\gcd(k_0 k_1 \dots k_n, k_{n+1}) = \gcd(a_n^2 - a_n + 1, a_n^2 + a_n + 1) = 1,$$

hence  $k_0, k_1, \dots, k_{n+1}$  are pairwise relatively prime. This completes the proof.

**Solution 3.** We have to find  $n$ , for which there exist positive integers  $a, b$  such that  $\gcd(a, b) = 1$  and

$$\frac{9n - 1}{n + 7} = \frac{a^2}{b^2}$$

From this relation we get

$$\frac{63n - 7}{n + 7} = \frac{7a^2}{b^2}$$

Thus

$$n = \frac{7a^2 + b^2}{9b^2 - a^2} = -7 + \frac{64b^2}{9b^2 - a^2}$$

Since  $\gcd(a, b) = 1$  it follows that  $\gcd(a^2, b^2) = 1$  and  $\gcd(9b^2 - a^2, b^2) = 1$ , so  $n$  is an integer if and only if  $9b^2 - a^2$  is a divisor of  $64$ . Also, since  $n$  is positive,  $9b^2 - a^2$  must be positive.

Now  $9b^2 - a^2 = (3b + a)(3b - a)$ . If  $a = b = 1$  then  $9b^2 - a^2 = 8$ ; otherwise,  $9b^2 - a^2 \geq 3b + a \geq 5$ , so  $9b^2 - a^2 \geq 8$ . So the possible values for  $9b^2 - a^2$  are  $8, 16, 32, 64$ . The factors  $3b + a$  and  $3b - a$  differ by a multiple of  $2$ , sum to a multiple of  $6$ , and satisfy  $3b + a > 3b - a$ , so the possibilities for  $(3b + a, 3b - a)$  are  $(4, 2), (16, 2)$ , and  $(8, 4)$ . The corresponding possibilities for  $(a, b)$  are  $(1, 1), (7, 3)$  and  $(2, 2)$ , but  $\gcd(2, 2) = 2 \neq 1$ . So, substituting the remaining pairs into  $n = \frac{7a^2 + b^2}{9b^2 - a^2}$  we get  $n = 1$  or  $n = 11$ .

**Solution 4.** Taking modulo  $5$ , we get

$$24a^2 + 1 \equiv b^2 \pmod{5} \text{ and } 25a^2 \equiv 0 \pmod{5}.$$

$$\text{Therefore, } -a^2 + 1 \equiv b^2 \pmod{5} \text{ or } a^2 + b^2 \equiv 1 \pmod{5}.$$

Since for any integer  $a$ ,  $a \equiv 0, 1, 2, 3, 4 \pmod{5}$  and so that  $a^2 \equiv 0, 1, 4 \pmod{5}$ , we will get the only possibility is that one of  $a^2, b^2$  is  $0$  and the other  $1 \pmod{5}$ .

Hence proved.

**Solution 5.** Here we have  $10000 \mid x^2 - x = x(x - 1)$ .

Since  $x$  and  $x - 1$  are co-prime, and  $10000 = 2^4 5^4$ , we have either  $16 \mid x$  and  $625 \mid x - 1$  or  $16 \mid x - 1$  and  $625 \mid x$ .

The 4-digit odd multiples of  $625$  are  $1875, 3135, 4375, 5625, 6875, 8125, 9375$ .

If  $1$  is added, only  $9375 + 1$  is divisible by  $16$ . If  $1$  is subtracted, then none is divisible by  $16$ . So  $x = 9376$  is the only answer.

**Solution 6.** Suppose that  $d \mid a$  and  $d \mid b$ . Then  $d$  divides any multiple of these two numbers and so divides  $\text{lcm}(a, b) = v$ .

Also,  $d \mid (a + b)$ . Hence  $d \mid \gcd(u, v)$ .

On the other hand, suppose that  $d \mid u$  and  $d \mid v$ . Let  $g = \gcd(d, a)$  and  $d = gh$ . We have that

$$v = \text{lcm}(a, b) = a \cdot \frac{b}{\gcd(a, b)}.$$

Since  $d$  divides  $v$ ,  $h$  divides  $d$  and  $\gcd(h, a) = 1$ , it follows that

$$h \mid \frac{b}{\gcd(a, b)}.$$

Now  $g \mid a + b$  and  $g \mid a$ , so  $g$  divides  $b = (a + b) - a$ . Also  $h \mid (a + b)$  and  $h \mid b$ , so  $h$  also divides  $a$ .

But, as  $\gcd(h, a) = 1$ ,  $h = 1$ . Hence  $d \mid a$ .

Similarly,  $d \mid b$ . Hence the pairs  $(a, b)$  and  $(u, v)$  have the same divisors and the result follows.

**Solution 7.** For any positive integer  $k$  we have,

$$5^k \equiv 5 \pmod{10}$$

$$2^{4k} \equiv 6 \pmod{10}$$

$$6^k \equiv 6 \pmod{10}$$

$$3^{4k} \equiv 1 \pmod{10}$$

Therefore,  $2^{2002} \equiv 6 \cdot 4 \equiv 4 \pmod{10}$ ,  $3^{2002} \equiv 1 \cdot 9 \equiv 9 \pmod{10}$ ,  $4^{2002} \equiv 6 \pmod{10}$

Hence  $a \equiv 4 + 9 + 6 + 5 \equiv 4 \pmod{10}$  and  $a^2 \equiv 6 \pmod{10}$ .

Now  $b = \frac{1}{2}(3^{2003} - 3)$ , but  $3^{2003} - 3 \equiv 7 - 3 \equiv 4 \pmod{10}$  and so  $b \equiv 2 \pmod{10}$

Therefore  $b^2 \equiv 4 \pmod{10}$

Finally,  $ab \equiv 4 \cdot 2 \equiv 8 \pmod{10}$ .

Hence the units digits of  $a^2$ ,  $b^2$  and  $ab$  are respectively 6, 4 and 8.

**Solution 8.** Let  $\sqrt{k^2 - pk} = n$ , where  $n \in \mathbb{N}$

Thus  $k^2 - pk - n^2 = 0$ , and  $k = \frac{p \pm \sqrt{p^2 + 4n^2}}{2}$ , which implies that  $p^2 + 4n^2$  is a perfect square.

Let  $p^2 + 4n^2 = m^2$ , where  $m \in \mathbb{N}$

So,  $(m - 2n)(m + 2n) = p^2$ .

Since  $p$  is a prime and  $p \geq 3$ , we have  $m - 2n = 1$ , and  $m + 2n = p^2$ .

Therefore,  $m = \frac{p^2 + 1}{2}$  and  $n = \frac{p^2 - 1}{4}$ .

So,  $k = \frac{p \pm m}{2} = \frac{2p \pm (p^2 + 1)}{4}$ .

Thus,  $k = \left(\frac{p+1}{2}\right)^2$  (The other value of  $k$  gives the same result.)

**Solution 9.** Let  $p$  be the smallest prime factor of  $n$ , and let  $d = n/p$ .

Then

$$\frac{np + p}{n + p} = \frac{p(n + 1)}{p(d + 1)} = \frac{n + 1}{d + 1}$$

By the given condition,  $\frac{n + 1}{d + 1}$  is an integer, therefore  $n + p$  divides  $np + p$ . Also  $n + p$  divides  $np + p^2$ , so it must divide the difference



$$(np + p^2) - (np + p) = p^2 - p.$$

Therefore,  $n + p \leq p^2 - p$ .

Thus  $n < p^2$ , so, dividing by  $p$ , we have  $d < p$ .

Suppose that  $d$  has some prime factor  $q$ .

Then  $q \leq d < p$ . On the other hand,  $q$  also divides  $n$ , and then the minimality of  $p$  gives  $q \geq p$ ,

which is a contradiction. So  $q$  cannot exist, and we conclude that  $d=1$ .

Thus  $n=p$ , i.e.,  $n$  is prime.

**Solution 10.** If  $(x, y)$  is a pair of integers that satisfies the inequality, then  $(-x, -y)$  is also such a pair, since

$$(-x)^2 + (-x)(-y) + (-y)^2 = x^2 + xy + y^2.$$

Again,  $(0, 0)$  is also a pair satisfying the inequality.

Thus, Every solution will be paired with a different solution, except for the one remaining solution  $(0, 0)$  which is paired with itself. This shows that the number of solutions is odd.

**Solution 11.** Let  $m$  denote the product of all primes  $p$  which are in  $P$ . Then we consider  $x = 2^{m+1}$ .

Now we prove that  $x$  satisfy the condition :  $x = 2^m + 2^m = (2^{\frac{m}{p}})^p + (2^{\frac{m}{p}})^p, \forall p$  which are in  $P$ .

So, it can be represented in the form ,  $a^p + b^p \forall p \in P$ . Next we will prove that the equation , $2^m = a^p + b^p$ , has no solution for which  $p$  is not in  $P$ .

**Case-I:**  $p = 2$ , then  $a = 2^{k_1}a_1$  and  $b = 2^{k_2}b_1$  where  $a, b$  are congruent to 1 modulo 2. It is easy to check that  $k_1 = k_2$  and  $a_1, b_1$  are congruent to 1 modulo 2.

**Case-II:**  $a = 2^{k_1}c, b = 2^{k_2}d$  where  $c, d$  are congruent to 1 modulo 2. Suppose  $a^p + b^p = 2^{m+1}$  then  $k_1 = k_2$ . Because  $\frac{c^p+d^p}{c+d}$  divides  $a^p + b^p$  and  $\frac{c^p+d^p}{c+d} = 1$  modulo 2. from the above we can infer that  $(c, d) = (1, 1)$  and it gives that  $a = b = 2^k$ . So  $2^{pk} = 2^m$  or we have that  $p$  divides  $m$  and so  $p \in P$ .

**Solution 12.** The key observation is that the equation can be rewritten as

$$\left(\frac{x}{2}\right)^2 = \left(\frac{z+y}{2}\right)\left(\frac{z-y}{2}\right).$$

Reading the equation  $x^2 + y^2 = z^2$  modulo 2, we see that both  $y$  and  $z$  are odd. Hence,  $\frac{z+y}{2}, \frac{z-y}{2}$ , and  $\frac{x}{2}$  are positive integers. We also find that  $\frac{z+y}{2}$  and  $\frac{z-y}{2}$  are relatively prime. Indeed, if  $\frac{z+y}{2}$  and  $\frac{z-y}{2}$  admits a common prime divisor  $p$ , then  $p$  also divides both  $y = \frac{z+y}{2} - \frac{z-y}{2}$  and  $\left(\frac{x}{2}\right)^2 = \left(\frac{z+y}{2}\right)\left(\frac{z-y}{2}\right)$ , which means that the prime  $p$  divides both  $x$  and

$y$ . This is a contradiction for  $\gcd(x, y) = 1$ . Now, applying the above lemma, we obtain

$$\left(\frac{x}{2}, \frac{z+y}{2}, \frac{z-y}{2}\right) = (pq, p^2, q^2)$$

for some positive integers  $p$  and  $q$  such that  $\gcd(p, q) = 1$ .

**Solution 13.** Assume to the contrary that there exists a bad triple  $(x, y, z)$  of positive integers such that  $x^4 + y^4 = z^2$ . Pick a bad triple  $(A, B, C) \in \mathcal{D}$  so that  $A^4 + B^4 = C^2$ . Letting  $d$  denote the greatest common divisor of  $A$  and  $B$ , we see that  $C^2 = A^4 + B^4$  is divisible by  $d^4$ , so that  $C$  is divisible by  $d^2$ . In the view of  $\left(\frac{A}{d}\right)^4 + \left(\frac{B}{d}\right)^4 = \left(\frac{C}{d^2}\right)^2$ , we find that  $(a, b, c) = \left(\frac{A}{d}, \frac{B}{d}, \frac{C}{d^2}\right)$  is also in  $\mathcal{D}$ , that is,

$$a^4 + b^4 = c^2.$$

Furthermore, since  $d$  is the greatest common divisor of  $A$  and  $B$ , we have  $\gcd(a, b) = \gcd\left(\frac{A}{d}, \frac{B}{d}\right) = 1$ . Now, we do the parity argument. If both  $a$  and  $b$  are odd, we find that  $c^2 \equiv a^4 + b^4 \equiv 1 + 1 \equiv 2 \pmod{4}$ , which is impossible. By symmetry, we may assume that  $a$  is even and that  $b$  is odd. Combining results, we see that  $a^2$  and  $b^2$  are relatively prime and that  $a^2$  is even. Now, in the view of  $(a^2)^2 + (b^2)^2 = c^2$ , we obtain

$$(a^2, b^2, c) = (2pq, p^2 - q^2, p^2 + q^2).$$

for some positive integers  $p$  and  $q$  such that  $\gcd(p, q) = 1$ . It is clear that  $p$  and  $q$  are of opposite parity. We observe that

$$q^2 + b^2 = p^2.$$

Since  $b$  is odd, reading it modulo 4 yields that  $q$  is even and that  $p$  is odd. If  $q$  and  $b$  admit a common prime divisor, then  $p^2 = q^2 + b^2$  guarantees that  $p$  also has the prime, which contradicts for  $\gcd(p, q) = 1$ . Combining the results, we see that  $q$  and  $b$  are relatively prime and that  $q$  is even. In the view of  $q^2 + b^2 = p^2$ , we obtain

$$(q, b, p) = (2mn, m^2 - n^2, m^2 + n^2).$$

for some positive integers  $m$  and  $n$  such that  $\gcd(m, n) = 1$ . Now, recall that  $a^2 = 2pq$ . Since  $p$  and  $q$  are relatively prime and since  $q$  is even, it guarantees the existence of the pair  $(P, Q)$  of positive integers such that

$$a = 2PQ, p = P^2, q = 2Q^2, \gcd(P, Q) = 1.$$

It follows that  $2Q^2 = 2q = 2mn$  so that  $Q = mn$ . Since  $\gcd(m, n) = 1$ , this guarantees the existence of the pair  $(M, N)$  of positive integers such

that

$$Q = MN, m = M^2, n = N^2, \gcd(M, N) = 1.$$

Combining the results, we find that  $P^2 = p = m^2 + n^2 = M^4 + N^4$  so that  $(M, N, P)$  is a bad triple. Recall the starting equation  $A^4 + B^4 = C^2$ . Now, let's summarize up the results what we did. The bad triple  $(A, B, C)$  produces a new bad triple  $(M, N, P)$ . However, we need to check that it is indeed new. We observe that  $P < C$ . Indeed, we deduce

$$P \leq P^2 = p < p^2 + q^2 = c = \frac{C}{d^2} \leq C.$$

In words, from a solution of  $x^4 + y^4 = z^2$ , we are able to find another solution with smaller positive integer  $z$ . The key point is that this reducing process can be repeated. Hence, it produces to an infinite sequence of strictly decreasing positive integers. However, it is clearly impossible. We therefore conclude that there exists no bad triple.

**Solution 14.** When  $4ab - 1$  divides  $(4a^2 - 1)^2$  for two distinct positive integers  $a$  and  $b$ , we say that  $(a, b)$  is a bad pair. We want to show that there is no bad pair. Suppose that  $4ab - 1$  divides  $(4a^2 - 1)^2$ . Then,  $4ab - 1$  also divides

$$b(4a^2 - 1)^2 - a(4ab - 1)(4a^2 - 1) = (a - b)(4a^2 - 1).$$

The converse also holds as  $\gcd(b, 4ab - 1) = 1$ . Similarly,  $4ab - 1$  divides  $(a - b)(4a^2 - 1)^2$  if and only if  $4ab - 1$  divides  $(a - b)^2$ . So, the original condition is equivalent to the condition

$$4ab - 1 \mid (a - b)^2.$$

This condition is symmetric in  $a$  and  $b$ , so  $(a, b)$  is a bad pair if and only if  $(b, a)$  is a bad pair. Thus, we may assume without loss of generality that  $a > b$  and that our bad pair of this type has been chosen with the smallest possible vales of its first element. Write  $(a - b)^2 = m(4ab - 1)$ , where  $m$  is a positive integer, and treat this as a quadratic in  $a$ :

$$a^2 + (-2b - 4ma)a + (b^2 + m) = 0.$$

Since this quadratic has an integer root, its discriminant

$$(2b + 4mb)^2 - 4(b^2 + m) = 4(4mb^2 + 4m^2b^2 - m)$$

must be a perfect square, so  $4mb^2 + 4m^2b^2 - m$  is a perfect square. Let his be the square of  $2mb + t$  and note that  $0 < t < b$ . Let  $s = b - t$ . Rearranging again gives:

$$\begin{aligned} 4mb^2 + 4m^2b^2 - m &= (2mb + t)^2 \\ m(4b^2 - 4bt - 1) &= t^2 \end{aligned}$$

$$\begin{aligned} m(4b^2 - 4b(b-s) - 1) &= (b-s)^2 \\ m(4bs - 1) &= (b-s)^2. \end{aligned}$$

Therefore,  $(b, s)$  is a bad pair with a smaller first element, and we have a contradiction.

**Solution 15.** We first evaluate  $f(n)$  for small  $n$ . It follows from  $f(1 \cdot 1) = f(1) \cdot f(1)$  that  $f(1) = 1$ . By the multiplicity, we get  $f(4) = f(2)^2 = 4$ . It follows from the inequality  $2 = f(2) < f(3) < f(4) = 4$  that  $f(3) = 3$ . Also, we compute  $f(6) = f(2)f(3) = 6$ . Since  $4 = f(4) < f(5) < f(6) = 6$ , we get  $f(5) = 5$ . We prove by induction that  $f(n) = n$  for all  $n \in \mathbb{N}$ . It holds for  $n = 1, 2, 3$ . Now, let  $n > 2$  and suppose that  $f(k) = k$  for all  $k \in \{1, \dots, n\}$ . We show that  $f(n+1) = n+1$ .

*Case 1.*  $n+1$  is composite. One may write  $n+1 = ab$  for some positive integers  $a$  and  $b$  with  $2 \leq a \leq b \leq n$ . By the inductive hypothesis, we have  $f(a) = a$  and  $f(b) = b$ . It follows that  $f(n+1) = f(a)f(b) = ab = n+1$ .

*Case 2.*  $n+1$  is prime. In this case,  $n+2$  is even. Write  $n+2 = 2k$  for some positive integer  $k$ . Since  $n \geq 2$ , we get  $2k = n+2 \geq 4$  or  $k \geq 2$ . Since  $k = \frac{n+2}{2} \leq n$ , by the inductive hypothesis, we have  $f(k) = k$ . It follows that  $f(n+2) = f(2k) = f(2)f(k) = 2k = n+2$ . From the inequality

$$n = f(n) < f(n+1) < f(n+2) = n+2$$

we conclude that  $f(n+1) = n+1$ . By induction,  $f(n) = n$  holds for all positive integers  $n$ .

**Solution 16.** First assume that  $p$  is prime. Now we consider  $n$  as  $n = ap+b$  where  $a$  is a non-negative integer and  $b$  an integer  $0 \leq b < p$ . Obviously,

$$\lfloor \frac{n}{p} \rfloor = \lfloor \frac{ap+b}{p} \rfloor \equiv a \pmod{p}.$$

Now let us calculate  $\binom{n}{p} \pmod{p}$ .

$$\begin{aligned} \binom{n}{p} &= \binom{ap+b}{p} \\ &= \frac{(ap+b) \cdot (ap+b-1) \cdots (ap+1) \cdot ap \cdot (ap-1) \cdots (ap+b-p+1)}{p \cdot (p-1) \cdots 2 \cdot 1} \\ &= \frac{a \cdot (ap+b) \cdot (ap+b-1) \cdots (ap+1) \cdot (ap-1) \cdots (ap+b-p+1)}{(p-1) \cdot (p-2) \cdots 2 \cdot 1} \end{aligned}$$

We denote this number by  $X$ .

We have  $X \equiv c \pmod{p}$  for some  $0 \leq c < p$ . Consequently taking modulo  $p$ , we have

$$c(p-1)! = X(p-1)! = a(ap+b) \cdots (ap+1)(ap-1) \cdots (ap+b-p+1)$$

All the numbers  $ap+b, \dots, ap+b+1-p$  (other than  $ap$ ) are relatively prime to  $p$  and obviously none differ more than  $p$  so they make a reduced residue system modulo  $p$ , meaning we have mod  $p$ ,

$$(p-1)! = (ap+b) \cdot (ap+b-1) \cdots (ap+1) \cdot (ap-1) \cdot (ap+b-p+1)$$

both sides of the equation being relatively prime to  $p$  so we can deduce  $X \equiv c \equiv a \pmod{p}$ . And finally  $\binom{n}{p} \equiv X \equiv a \equiv \lfloor \frac{n}{p} \rfloor \pmod{p}$ .

To complete the other part of the theorem we must construct a counterexample for every composite number  $p$ . If  $p$  is composite we can consider it as  $q^x \cdot k$  where  $q$  is some prime factor of  $p$ ,  $x$  it's exponent and  $k$  the part of  $p$  that is relatively prime to  $q$  ( $x$  and  $k$  cannot be simultaneously 1 or  $p$  is prime). The following  $n = p + q = q^x k + q$  will make a counter example. We have:

$$\binom{p+q}{p} = \binom{p+q}{q} = \frac{(q^x k + q)(q^x k + q - 1) \cdots (q^x k + 1)}{q!}$$

Which after simplifying the fraction equals:  $(q^{x-1}k+1) \frac{(q^x k + q - 1) \cdots (q^x k + 1)}{(q-1)!}$ . Similarly as above we have  $(q^x k + q - 1) \cdots (q^x k + 1) = (q-1)! \not\equiv 0$  modulo  $q^x$  therefore,

$$\frac{(q^x k + q - 1) \cdots (q^x k + 1)}{(q-1)!} \equiv 1 \pmod{q^x}$$

and

$$\binom{p+q}{p} \equiv q^{x-1}k + 1 \pmod{q^x}.$$

On the other hand obviously,

$$\lfloor \frac{q^x k + q}{q^x k} \rfloor \equiv 1 \pmod{q^x}.$$

And since  $q^{x-1}k + 1$  can never be equal to 0 modulo  $q^x$  we see that

$$\binom{p+q}{p} \not\equiv \lfloor \frac{p+q}{p} \rfloor \pmod{q^x}$$

consequently also incongruent modulo  $p = q^x k$ .

**Solution 17.** If  $x = 1$  the sequence is obviously periodic for any modulo  $m$ .

Now we assume that the sequence is periodic for a fixed  $x$  and arbitrary  $m$ . We note that

$$\binom{n}{x+1} = \sum_{i=1}^{n-1} \binom{i}{x}.$$

Let  $k$  be the length of a period of sequence  $a_n = \binom{n}{x} \pmod{m}$ , meaning  $\binom{n+k}{x} \equiv \binom{n}{x} \pmod{m}$ .

Therefore  $\sum_{i=1}^k \binom{i}{x} \equiv c \pmod{m}$  for some  $c$  and consequently  $\sum_{i=n+1}^{n+mk} \binom{i}{x} = mc = 0 \pmod{m}$  for every integer  $n$ . All that is now required is another calculation (the second equality from the right is modulo  $m$ ):

$$\binom{n+mk}{x+1} = \sum_{i=1}^{n+mk-1} \binom{i}{x} = \sum_{i=1}^{n-1} \binom{i}{x} + \sum_{i=n}^{n+mk-1} \binom{i}{x} = \sum_{i=1}^{n-1} \binom{i}{x} = \binom{n}{x+1}$$

This now shows that sequence  $b_n = \binom{n}{x+1} \pmod{m}$  is also periodic for every modulo  $m$  which completes the induction and yields the desired result.

**Solution 18.** A sequence  $a_n = \binom{n}{m} \pmod{m}$  where  $m = \prod_{i=1}^k p_i^{b_i}$  starts with  $m$  zeroes (we start with  $a_0$ ). Now let us see when is the next time we have  $m$  consecutive zeroes in the sequence  $a_n$ . We assume this happens at some natural number  $n$ , that is

$$\binom{n}{m} \equiv \binom{n+1}{m} \equiv \dots \equiv \binom{n+m-1}{m} \equiv 0 \pmod{m}.$$

Let  $p$  be a prime dividing  $m$  and  $b$  be it's exponent in the prime factorization of  $m$ . We have  $\binom{n+i}{m} \equiv 0 \pmod{p^b}$  for  $0 \leq i < m$ .

Obviously the exponent of  $p$  in prime factorization of  $m!$  is

$$\vartheta_p(m) = \sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor = \sum_{i=1}^k \left\lfloor \frac{m}{p^i} \right\rfloor,$$

where  $k$  is the last summand different to zero and  $k = \lfloor \log_p m \rfloor$ .

Among numbers  $n+1, n+2, \dots, n+m$  there exist one that is divisible by  $p^k$  (there are  $m$  consecutive numbers and  $m \geq p^k$ ). We denote this number by  $x$ . We have,

$$\binom{x-1}{m} = \frac{(x-1)(x-2)\dots(x-m)}{m!}.$$

Since we have  $-(x-i) \equiv i \pmod{p^j}$  for all  $1 \leq i < m$  and  $1 \leq j \leq k$ , so there are same number of numbers divisible by  $p^j$  in  $(x-1)(x-2)\dots(x-m)$  as in  $m!$  for  $1 \leq j \leq k$ .

On the other hand we have  $\binom{x-1}{m} \equiv 0 \pmod{p^b}$  (since  $x-1$  is one of the numbers  $n, n+1, \dots, n+m-1$ ). Of  $m$  consecutive integers obviously only one can be divisible by  $p^j$  if  $j < k$ . Therefore if we

want the numerator of  $\binom{x-1}{m}$  to have exponent of  $p$  for  $b$  larger than the denominator (that is in order to have  $\binom{x-1}{m} \equiv 0 \pmod{p^b}$ ) we need one of the numbers of the nominator to be divisible by  $p^{\lfloor \log_p m \rfloor + a}$ . Denote this number by  $y$ .

We assume  $y \neq n$ . Than either  $y + m$  (if  $y < n$ ) or  $y - 1$  (if  $y > n$ ) are in the set  $n, n + 1 \dots n + m - 1$ . This means that

$$\binom{y + m}{m} \equiv 0 \pmod{p^b}.$$

(The other case is very similar and uses the same argument.)

However that is imposible since  $y \equiv 0 \pmod{p^{k+1}}$  meaning the exponent of  $p$  in prime factorization of  $(y + m)(y + m - 1) \dots (y + 1)$  is the same as in prime factorization of  $m!$  or in other words that  $\binom{y+m}{m}$  is relatively prime to  $p$ . We reached a contradiction which means  $y = n$ .

The same argument will work for any arbitrary prime number dividing  $m$ . That means for every prime number  $p$  dividing  $m$  (infact  $p^b \mid m$ ) we need  $n$  to be divisible by  $p^{\lfloor \log_p m \rfloor + b}$ , therefore the length of the period of the sequence,  $a_n$  must be a multiple of the number  $\prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor + b_i}$ .

All that remains is to show that this infact is the length of the period. We need to prove that for every natural number  $n$  we have

$$\binom{n}{m} \equiv \binom{n + l(m)}{m} \pmod{m},$$

where

$$l(m) = \prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor + b_i}.$$

Because of some basic properties of congruences ( $a \equiv b \pmod{m}$  equivalent to  $ax \equiv bx \pmod{m}$  if  $\gcd(m, x) = 1$ ), it is enough to show that,

$$\frac{\prod_{i=0}^{m-1} (n - i)}{\prod_{i=1}^k p_i^{\vartheta_{p_i}(m)}} \equiv \frac{\prod_{i=0}^{m-1} (n + l(m) - i)}{\prod_{i=1}^k p_i^{\vartheta_{p_i}(m)}} \pmod{m}.$$

Among the numbers  $n, n - 1 \dots n - m + 1$  there are atleast  $\lfloor \frac{n}{p^l} \rfloor$  that are divisible by  $p^l$  for every positive integer  $l$  and any prime divisor  $p$  of  $m$ .

This is because

$$\prod_{i=0}^{m-1} (n - i) = \frac{n!}{(n - m)!}$$

and

$$\vartheta_p(a + b) \geq \vartheta_p(a) + \vartheta_p(b).$$

The fraction  $\frac{\prod_{i=0}^{m-1} (n-i)}{\prod_{i=1}^k p_i^{\nu_{p_i}(m)}}$  can therefore be simplified in such a way that no number of the product  $\prod_{i=0}^{m-1} (n-i)$  is divided by  $p$  on exponent greater than  $\lfloor \log_p m \rfloor$ .

In other words the fraction  $\frac{\prod_{i=0}^{m-1} (n-i)}{\prod_{i=1}^k p_i^{\nu_{p_i}(m)}}$  can be simplified as  $\prod_{i=0}^{m-1} \frac{n-i}{\prod_{j=1}^k p_j^{c_j}}$  where for each  $j, i$  we have  $n-i$  divisible by  $p_j^{c_j}$  and  $c_j \leq \lfloor \log_{p_j} m \rfloor$  (each factor is an integer).

But then since for every  $j$  we have  $\prod_{j=1}^k p_j^{c_j}$  divides  $\prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor}$  and since  $m \cdot \prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor} = l(m)$  we have for every  $i \pmod m$ ,

$$\frac{n + l(m) - i}{\prod_{j=1}^k p_j^{c_j}} = \frac{n - i}{\prod_{j=1}^k p_j^{c_j}} + \frac{l(m)}{\prod_{j=1}^k p_j^{c_j}} = \frac{n - i}{\prod_{j=1}^k p_j^{c_j}} + t \cdot m = \frac{n - i}{\prod_{j=1}^k p_j^{c_j}}$$

This completes the result and hence the length of the minimal period of the sequence,  $a_n$  is

$$l(m) = \prod_{i=1}^k p_i^{\lfloor \log_{p_i} m \rfloor + b_i}.$$

**Solution 19.** Note that if  $n$  is relatively prime to  $m$  than so is  $n - m$ . We have

$$\binom{n}{m} = \binom{n-1}{m} \cdot \frac{n}{n-m} \pmod m$$

which is equivalent to

$$(n-m) \cdot \binom{n}{m} = n \cdot \binom{n-1}{m} \pmod m$$

which is further equivalent to

$$\binom{n}{m} = \binom{n-1}{m} \pmod m$$

because  $n = n - m \pmod m$  and both are relatively prime to  $m$ .

**Solution 20.** We have

$$\binom{l(m) - 1 - k}{m} = \frac{(l(m) - 1 - k)(l(m) - 1 - k - 1) \dots (l(m) - 1 - m - k + 1)}{m!}$$

and because there are an even number ( $m$ ) of factors we can multiply each one by  $-1$  and still have the same number. So,

$$\binom{l(m) - 1 - k}{m} = \frac{(k + 1 - l(m))(k + 2 - l(m)) \dots (k + m - l(m))}{m!}$$



which is precisely  $\binom{k+m-l(m)}{m}$  and is by the previous theorem equal to  $\binom{m+k}{m} \pmod{m}$ .

**Solution 21.** Intuitively, the sequence being required to be strictly monotonic points that it will eventually grow rather fast. This is a hint to the solution. Note that if  $n$  is even, then the set of divisors of  $n^2 + 1$  can be partitioned into pairs  $\left\{d, \frac{n^2+1}{d}\right\}$ , where  $d < \frac{n^2+1}{d}$ . Clearly  $d$  is odd and less than  $n$ . Hence we have at most  $\frac{n}{2}$  pairs, consequently  $d(n^2 + 1) \leq n$ .

Assuming to the contrary that the sequence becomes strictly monotonic starting with an  $N$ , it's obvious that it must be increasing (otherwise  $d(n^2 + 1)$  would be forced to take negative values from some point  $n > N$  onwards). Note that since  $n^2 + 1$  is not a perfect square for any  $n > 0$ , hence  $d(n^2 + 1)$  is an even number for every positive integer  $n$ . Since  $d(n^2 + 1)$  is strictly monotonic for  $n \geq N$ , we deduce

$$d((n+1)^2 + 1) \geq d(n^2 + 1) + 2.$$

A straightforward induction proves that

$$d((n+k)^2 + 1) \geq d(n^2 + 1) + 2k.$$

By the inequality established in the beginning of the solution, for  $N+t$  even we obtain the inequalities

$$N+t > d((N+t)^2 + 1) \geq d(N^2 + 1) + 2t,$$

or

$$N > d(N^2 + 1) + t$$

for any  $t > 0$  which is impossible, since the rest of the terms of the inequality are constant.

**Solution 22.** Note that since the sequence is not required to be strictly monotonic, we cannot infer that it will grow very fast, so the argument used at (a) fails. We will prove the following generalization:

**Claim 1.** Let  $t$  and  $m$  be two positive integers. Then the sequence  $d((n^2+m^2)^t)$  does not become monotonic from any given point onwards.

Suppose, to the contrary, that from some point onwards, the sequence becomes monotonic.

We will firstly show that it must be increasing. Indeed, take a prime  $p$  of the form  $4k+1$ . Clearly  $-1$  is a quadratic residue  $\pmod{p}$ , hence so is  $-m^2$ , so there is an integer  $r$  so that  $p \mid r^2+m^2$ . Take now  $d$  different primes  $s_1, \dots, s_d$  of the form  $4u+1$  and let  $r_i \in \mathbb{Z}$  so that  $s_i \mid r_i^2+m^2$ . Using the Chinese Remainder Theorem there is an integer  $N$ , so that

$N \equiv r_i \pmod{s_i}$ , for every  $i = 1, \dots, d$ . Then  $N^2 + m^2 \equiv r_i^2 + m^2 \pmod{s_i}$ , hence  $s_1 \dots s_d \mid N^2 + m^2$ . This implies that  $d((N^2 + m^2)^t)$  is unbounded, consequently it must be increasing from some point  $x_0$  onwards.

For shortness of notations let  $f_n = f(n) = d((n^2 + m^2)^t)$ . We will use the following very simple result.

**Lemma 2.1.**  $\gcd(a^2 + m^2, (a-1)^2 + m^2) = 1$  if  $\gcd(2a-1, 4m^2+1) = 1$ .

*Proof.* Let  $\gcd(2a-1, 4m^2+1) = 1$  and suppose there is a prime  $p$  dividing both  $a^2 + m^2$  and  $(a-1)^2 + m^2$ . By subtraction, we obtain  $p \mid 2a-1$ . Then  $2a \equiv 1 \pmod{p}$ , so  $4a^2 \equiv 1 \pmod{p}$ , or  $4a^2 + 4m^2 \equiv 1 + 4m^2 \pmod{p}$ . Since  $p \mid 4(a^2 + m^2)$  we obtain  $0 \equiv 1 + 4m^2 \pmod{p}$ , contradicting  $\gcd(2a-1, 4m^2+1) = 1$ .  $\square$

Take  $x > x_0$  so that  $\gcd(2x-1, 1+4m^2) = 1$ . Then from Lemma 1 and the identity  $[x^2 + m^2][(x-1)^2 + m^2] = (x^2 - x + m^2)^2 + m^2$  we get the inequality  $f_{x-1}f_x \leq f_{x^2-x+m^2}$ , since  $d(uv) = d(u) \cdot d(v)$  if  $\gcd(u, v) = 1$ .

We now state the following result, which we are going to use a bit later.

**Lemma 2.2.** Let  $M$  be an integer. Then there exists a positive integer  $\lambda$  so that the polynomial  $h(x) = 4x^2 - \lambda$  satisfies

$$\gcd(2h(x) + 1, M) = 1, \forall x \in \mathbb{Z}$$

*Proof.* Since  $2h(x)+1$  is odd, we need only prove the lemma for odd  $M$ . So assume  $M$  is odd and let  $\{b_1, \dots, b_s\}$  be the set of prime divisors of  $M$ . We are looking for  $\lambda$  so that  $b_i \nmid 2h(x) + 1 = 8x^2 - 2\lambda + 1, \forall i = \overline{1, s}$ . Since  $b_i$ 's are odd, the last condition is equivalent to  $b_i \nmid (4x)^2 - (4\lambda - 2)$ . It is enough to find a  $\lambda$  so that  $4\lambda - 2$  is a quadratic nonresidue  $\pmod{b_i}$ . For every prime  $b_i$  there exists a quadratic non-residue  $r_i$  (actually there are  $\frac{b_i-1}{2}$  of them). We will apply once again the Chinese Remainder Theorem in the following way:

We are looking for an integer  $L$  satisfying the following system of equations:

$$L \equiv r_i \pmod{b_i}, \forall i = \overline{1, s}$$

$$L \equiv 2 \pmod{4}$$

and take  $\lambda = \frac{L-2}{4}$ . Clearly we can assume  $\lambda > 0$ .  $\square$

Let's continue with the problem. Take  $M = 1 + 4m^2$  in Lemma 2 to obtain such  $\lambda$  and  $h(x)$ . Using the monotonicity of  $f$  we deduce the chain of inequalities

$$f_{x-1}^2 \leq f_{x-1}f_x \leq f_{x^2-x+m^2} \leq f_{4(x-1)^2-\lambda},$$

for sufficiently large  $x > x_0$ . Here, we may also assume that  $x_0$  is sufficiently large so that  $x > x_0$  guarantees that  $h(x) > x_0$ . Note that the inequality  $f_{x-1}f_x \leq f_{x^2-x+m^2}$  provides another proof that if  $f$  is monotonic, then it must be increasing. Hence  $f_q^2 \leq f_{h(q)}$ , where  $q = x - 1 \geq x_0$ , and  $\gcd(2q + 1, 1 + 4m^2) = 1$ . Because by Lemma 2 we have  $\gcd(2h(q) + 1, 4m^2 + 1) = 1$  we further get  $f(q)^4 \leq \{f(h(q))\}^2 \leq f[h(h(q))]$ . By an easy induction we obtain the inequalities

$$f(q)^{2^k} \leq f \left( \underbrace{h(h(\dots h(q)\dots))}_{k \text{ times}} \right) \leq f \left[ (4q)^{2^k} \right].$$

Here we have iteratively used the fact that  $h(z) < 4z^2$ . We are going now to summarize the obtained results. Let  $c = f(q)$  and define  $g(z)$  to be the positive integer satisfying

$$(4q)^{2^{g(z)}} \leq z < (4q)^{2^{g(z)+1}}.$$

We easily obtain  $g(z) = \lfloor \log_2 \lfloor \log_{4q} z \rfloor \rfloor$ . Then the above inequality and the monotonicity of  $f$  implies

$$c^{2^{g(z)}} \leq f(z)$$

for sufficiently large  $z$ . With this, we have found a lower estimate for  $f(z)$ .

Let's find an upper estimate for  $f(x)$  which would contradict, for large enough  $x$  the lower estimate obtained above. For this, let  $(p_i)_{i \geq 1}$  be the sequence of prime numbers, not containing the prime divisors of  $m$ .

Let's take a closer look at  $f(p_1 \dots p_k)$ . Let  $(p_1 \dots p_k)^2 + m^2 = \prod_{i=1}^s q_i^{\alpha_i}$ .

Using divisibility arguments, we have  $q_i > p_j$  for all  $i = \overline{1, s}$  and  $j = \overline{1, k}$ . This clearly implies  $\sum_{i=1}^s \alpha_i \leq 2k$ . Note that

$$f(p_1 \dots p_k) = d \left( [(p_1 \dots p_k)^2 + m^2]^t \right) = (t\alpha_1 + 1) \dots (t\alpha_s + 1) =_{\text{def}} h(\alpha_1, \dots, \alpha_s)$$

Using the already stated inequality  $\sum_{i=1}^s \alpha_i \leq 2k$  we will prove that  $h(\alpha_1, \dots, \alpha_s) \leq (t+1)^{2k}$ .

Indeed, note that if  $a > 1$  then  $(t+1)(t(a-1)+1) \geq ta+1$ . Hence if there is some  $\alpha_i > 1$ , Without Loss Of Generality,  $\alpha_1 > 1$ , we have  $h(\alpha_1, \alpha_2, \dots, \alpha_s) \leq h(\alpha_1 - 1, \alpha_2, \dots, \alpha_s, 1)$ . By repeated applications of this inequality until  $\alpha_i = 1$ , for all  $i$ , we obtain the following inequality

$$f(p_1 \cdots p_k) = h(\alpha_1, \dots, \alpha_s) \leq h\left(\underbrace{1, 1, \dots, 1}_{\sum \alpha_i}\right) \leq (t+1)^{\sum \alpha_i} \leq (t+1)^{2k} = T^k,$$

where  $T = (t+1)^2$ . Define now the function  $l(x)$  to be equal  $v+1$ , where  $v$  is the unique positive integer for which  $p_1 \dots p_v < x \leq p_1 \dots p_{v+1}$ . Using once again the monotonicity of  $f$ , we establish the following upper bound for the function  $f$ :

$$f(x) \leq f(p_1 \dots p_{l(x)}) \leq T^{l(x)}$$

Now, since  $g(x) = \lfloor \log_2 \lfloor \log_{4q} x \rfloor \rfloor$ , we have  $g(x) > \log_2 \lfloor \log_{4q} x \rfloor - 1$ , hence

$$2^{g(x)} > 2^{\log_2 \lfloor \log_{4q} x \rfloor - 1} = \frac{1}{2} \lfloor \log_{4q} x \rfloor.$$

It thus follows that

$$T^{l(x)} \geq f(x) \geq c^{2^{g(x)}} > \sqrt{c}^{\lfloor \log_{4q} x \rfloor}.$$

By the fact that  $f$  is unbounded, we can choose  $c$  as large as we want, hence we can assume  $\sqrt{c} > T^2$ . Then, for reaching a contradiction, we will show that  $l(x) < 2 \lfloor \log_{4q} x \rfloor$  for large enough  $x$ . Since  $1 + \log_{4t} x < -2 + 2 \log_{4t} x < 2 \lfloor \log_{4t} x \rfloor$  for  $\log_{4q} x > 3$ , it is sufficient to prove  $l(x) - 1 < \log_{4q} x$  for large enough  $x$ . The last inequality is equivalent to  $(4q)^{l(x)-1} < x$ . Recall that  $4q$  is a constant value. We find that the primes grow very fast so that the inequality  $(4q)^{l(x)-1} < p_1 \dots p_{l(x)-1}$  holds for large enough  $x$ . By the definition of  $l(x)$ , we have then, indeed,  $p_1 \dots p_{l(x)-1} < x$ , obtaining  $l(x) - 1 < \log_{4q} x$ , what we wanted.

**Solution 23.** Let  $x_n = a^n - b^n$ . We are given that  $x_n \in \mathbb{Z}$  for all  $n \in \mathbb{N}$ .

We can easily deduce that  $a, b$  are rational:  $\frac{x_2 \pm x_1}{2} = \frac{a+b \pm (a-b)}{2} = a, b$ . Assume, for contradiction's sake, that  $a$  is not an integer. We'll have  $a = \frac{p}{q}$ ,  $(p, q) = 1$ , and  $|q| > 1$ . There exist  $m \in \mathbb{N}_0$  such that  $q^m \mid a -$

$b, q^{m+1} \mid a - b$ . We have  $\left(x_1 + \frac{p}{q}\right)^n - \left(\frac{p}{q}\right)^n = x_n$ , or equivalently:

$$(2) \quad (x'q^{m+1} + p)^n - p^n = q^n x_n$$

For a suitable integer  $x'$ . Now we'll use the binomial theorem and divide by  $q^{m+1}$ :

$$(3) \quad \sum_{i=0}^{n-1} \binom{n}{i} p^i x'^{m-i} q^{(m+1)(n-i-1)} = q^{n-m-1} x_n.$$

Looking (mod  $q$ ), for  $n > m + 1$ , we see that:  $x'p^{n-1}n \equiv 0 \pmod{q}$ . Exploiting the fact that  $(q, p) = 1$  and taking  $n = (m + 2) \mid q \mid + 1$ , yields  $q \mid x'$ , a contradiction. Hence,  $q = \pm 1$  and  $a$  is an integer.  $b = a - x_1$  is thus also an integer.

We leave the rest of the problems for the reader to solve. Hints are given below.

**Solution 24.** In this problem, there are several ways to work with. Perhaps the easiest method would be to use congruence properties of integers.

**Solution 25.** This problem can be tricky, and uses some advanced techniques. However, the readers are advised to give it a try.

**Solution 26.** De-Polignac's formula would be useful in this case.

**Solution 27.** A similar problem was solved earlier, follow that problem.

**Solution 28.** Use properties of binomial coefficients judiciously along with congruence properties.

**Solution 29.** This problem needs some ingenuity, and is solved using ideas from set theory along with various number theoretic techniques.

**Solution 30.** This is a fairly straightforward problem, and uses only elementary number theory.

## 3. INEQUALITIES

Each problem that I solved became a rule, which served afterwards to solve other problems.

- R. Descartes

## 3.1. Preliminaries.

**Theorem 3.1. (Schur)** *Let  $x, y, z$  be nonnegative real numbers. For any  $r > 0$ , we have*

$$\sum_{\text{cyclic}} x^r (x - y)(x - z) \geq 0.$$

**Theorem 3.2. (Muirhead)** *Let  $a_1, a_2, a_3, b_1, b_2, b_3$  be real numbers such that*

$$a_1 \geq a_2 \geq a_3 \geq 0, b_1 \geq b_2 \geq b_3 \geq 0, a_1 \geq b_1, a_1 + a_2 \geq b_1 + b_2, a_1 + a_2 + a_3 = b_1 + b_2 + b_3.$$

*Let  $x, y, z$  be positive real numbers. Then, we have  $\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} \geq \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}$ .*

**Theorem 3.3. (The Cauchy-Schwarz inequality)** *Let  $a_1, \dots, a_n, b_1, \dots, b_n$  be real numbers. Then,*

$$(a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2) \geq (a_1 b_1 + \dots + a_n b_n)^2.$$

**Theorem 3.4. (AM-GM inequality)** *Let  $a_1, \dots, a_n$  be positive real numbers. Then, we have*

$$\frac{a_1 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \dots a_n}.$$

**Theorem 3.5. (Weighted AM-GM inequality)** *Let  $\omega_1, \dots, \omega_n > 0$  with  $\omega_1 + \dots + \omega_n = 1$ . For all  $x_1, \dots, x_n > 0$ , we have*

$$\omega_1 x_1 + \dots + \omega_n x_n \geq x_1^{\omega_1} \dots x_n^{\omega_n}.$$

**Theorem 3.6. (Hölder's inequality)** Let  $x_{ij}$  ( $i = 1, \dots, m, j = 1, \dots, n$ ) be positive real numbers. Suppose that  $\omega_1, \dots, \omega_n$  are positive real numbers satisfying  $\omega_1 + \dots + \omega_n = 1$ . Then, we have

$$\prod_{j=1}^n \left( \sum_{i=1}^m x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^m \left( \prod_{j=1}^n x_{ij}^{\omega_j} \right).$$

**Theorem 3.7. (Power Mean inequality)** Let  $x_1, \dots, x_n > 0$ . The power mean of order  $r$  is defined by

$$M_{(x_1, \dots, x_n)}(0) = \sqrt[n]{x_1 \cdots x_n}, \quad M_{(x_1, \dots, x_n)}(r) = \left( \frac{x_1^r + \dots + x_n^r}{n} \right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then,  $M_{(x_1, \dots, x_n)} : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and monotone increasing.

**Theorem 3.8. (Majorization inequality)** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function. Suppose that  $(x_1, \dots, x_n)$  majorizes  $(y_1, \dots, y_n)$ , where  $x_1, \dots, x_n, y_1, \dots, y_n \in [a, b]$ . Then, we obtain

$$f(x_1) + \dots + f(x_n) \geq f(y_1) + \dots + f(y_n).$$

**Theorem 3.9. (Bernoulli's inequality)** For all  $r \geq 1$  and  $x \geq -1$ , we have

$$(1 + x)^r \geq 1 + rx.$$

**Definition 3.1. (Symmetric Means)** For given arbitrary real numbers  $x_1, \dots, x_n$ , the coefficient of  $t^{n-i}$  in the polynomial  $(t + x_1) \cdots (t + x_n)$  is called the  $i$ -th elementary symmetric function  $\sigma_i$ . This means that

$$(t + x_1) \cdots (t + x_n) = \sigma_0 t^n + \sigma_1 t^{n-1} + \dots + \sigma_{n-1} t + \sigma_n.$$

For  $i \in \{0, 1, \dots, n\}$ , the  $i$ -th elementary symmetric mean  $S_i$  is defined by

$$S_i = \frac{\sigma_i}{\binom{n}{i}}.$$

**Theorem 3.10.** Let  $x_1, \dots, x_n > 0$ . For  $i \in \{1, \dots, n\}$ , we have

- (1) (Newton's inequality)  $\frac{S_i}{S_{i+1}} \geq \frac{S_{i-1}}{S_i}$ ,
- (2) (Maclaurin's inequality)  $S_i^{\frac{1}{i}} \geq S_{i+1}^{\frac{1}{i+1}}$ .

**Theorem 3.11. (Rearrangement inequality)** *Let  $x_1 \geq \cdots \geq x_n$  and  $y_1 \geq \cdots \geq y_n$  be real numbers. For any permutation  $\sigma$  of  $\{1, \dots, n\}$ , we have*

$$\sum_{i=1}^n x_i y_i \geq \sum_{i=1}^n x_i y_{\sigma(i)} \geq \sum_{i=1}^n x_i y_{n+1-i}.$$

**Theorem 3.12. (Chebyshev's inequality)** *Let  $x_1 \geq \cdots \geq x_n$  and  $y_1 \geq \cdots \geq y_n$  be real numbers. We have*

$$\frac{x_1 y_1 + \cdots + x_n y_n}{n} \geq \left( \frac{x_1 + \cdots + x_n}{n} \right) \left( \frac{y_1 + \cdots + y_n}{n} \right).$$

**Theorem 3.13. (Hölder's inequality)** *Let  $x_1, \dots, x_n, y_1, \dots, y_n$  be positive real numbers. Suppose that  $p > 1$  and  $q > 1$  satisfy  $\frac{1}{p} + \frac{1}{q} = 1$ . Then, we have*

$$\sum_{i=1}^n x_i y_i \leq \left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n y_i^q \right)^{\frac{1}{q}}$$

**Theorem 3.14. (Minkowski's inequality)** *If  $x_1, \dots, x_n, y_1, \dots, y_n > 0$  and  $p > 1$ , then*

$$\left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n y_i^p \right)^{\frac{1}{p}} \geq \left( \sum_{i=1}^n (x_i + y_i)^p \right)^{\frac{1}{p}}$$



## 3.2. Problems.

**Problem 31.** Let  $a, b, c$  be the lengths of the sides of a triangle. Suppose that  $u = a^2 + b^2 + c^2$  and  $v = (a + b + c)^2$ . Prove that

$$\frac{1}{3} \leq \frac{u}{v} < \frac{1}{2}$$

and that the fraction  $1/2$  on the right cannot be replaced by a smaller number.

**Problem 32.** Let  $a$  and  $b$  be positive real numbers. Prove that the inequality

$$\frac{(a+b)^3}{a^2b} \geq \frac{27}{4}$$

holds. When does equality hold?

**Problem 33.** Suppose  $x$  and  $y$  are positive real numbers such that  $x + 2y = 1$ . Prove that

$$\frac{1}{x} + \frac{2}{y} \geq \frac{25}{1 + 48xy^2}.$$

**Problem 34.** Let  $a_1, a_2, \dots, a_n$  be distinct positive integers. Prove that

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + \dots + \frac{a_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}.$$

**Problem 35.** Let  $x, y, z$  be positive real numbers and  $xyz \geq 1$ . Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0$$

**Problem 36.** Prove that, for every positive integer  $n$ :

$$\frac{1}{11} + \frac{2}{21} + \frac{3}{31} + \dots + \frac{n}{10n+1} < \frac{n}{10}$$

**Problem 37.** Let  $a, b, u, v$  be nonnegative numbers. Suppose that  $a^5 + b^5 \leq 1$  and  $u^5 + v^5 \leq 1$ . Prove that

$$a^2u^3 + b^2v^3 \leq 1.$$

**Problem 38.** Let  $n$  be a positive integer and  $x \neq 0$ . Prove that

$$(1+x)^{n+1} \geq \frac{(n+1)(n+1)}{n^n}x.$$

**Problem 39.** Prove that

$$(\sqrt{n-1} + \sqrt{n} + \sqrt{n+1})^2 < 9n.$$

**Problem 40.** Suppose that  $a_1 < a_2 < \dots < a_n$ . Prove that

$$a_1a_2^4 + a_2a_3^4 + \dots + a_na_1^4 \geq a_2a_1^4 + a_3a_2^4 + \dots + a_1a_n^4.$$

**Problem 41.** Let  $a, b, c$  be positive real numbers with  $ab + bc + ca = abc$ . Prove that

$$\frac{a^4 + b^4}{ab(a^3 + b^3)} + \frac{b^4 + c^4}{bc(b^3 + c^3)} + \frac{c^4 + a^4}{ca(c^3 + a^3)} \geq 1.$$

**Problem 42.** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove that

$$\sqrt{a + b - c} + \sqrt{b + c - a} + \sqrt{c + a - b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

**Problem 43.** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$\frac{\sqrt{b + c - a}}{\sqrt{b} + \sqrt{c} - \sqrt{a}} + \frac{\sqrt{c + a - b}}{\sqrt{c} + \sqrt{a} - \sqrt{b}} + \frac{\sqrt{a + b - c}}{\sqrt{a} + \sqrt{b} - \sqrt{c}} \leq 3.$$

**Problem 44.** Let  $ABC$  be a triangle. Prove that

$$\sin 3A + \sin 3B + \sin 3C \leq \frac{3\sqrt{3}}{2}.$$

**Problem 45.** (Chebyshev's Inequality) Let  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  be two monotone increasing sequences of real numbers:

$$x_1 \leq \dots \leq x_n, \quad y_1 \leq \dots \leq y_n.$$

Then, we have the estimation

$$\sum_{i=1}^n x_i y_i \geq \frac{1}{n} \left( \sum_{i=1}^n x_i \right) \left( \sum_{i=1}^n y_i \right).$$

**Problem 46.** Let  $a_1, \dots, a_n, b_1, \dots, b_n$  be positive real numbers such that  $a_1 + \dots + a_n = b_1 + \dots + b_n$ . Show that

$$\frac{a_1^2}{a_1 + b_1} + \dots + \frac{a_n^2}{a_n + b_n} \geq \frac{a_1 + \dots + a_n}{2}.$$

**Problem 47.** Let  $a, b, c, d \geq 0$  with  $ab + bc + cd + da = 1$ . show that

$$\frac{a^3}{b + c + d} + \frac{b^3}{c + d + a} + \frac{c^3}{d + a + b} + \frac{d^3}{a + b + c} \geq \frac{1}{3}.$$

**Problem 48.** (Weitzenböck's Inequality) Let  $a, b, c$  be the lengths of a triangle with area  $S$ . Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

**Problem 49.** (Hadwiger-Finsler Inequality) For any triangle  $ABC$  with sides  $a, b, c$  and area  $F$ , the following inequality holds.

$$2ab + 2bc + 2ca - (a^2 + b^2 + c^2) \geq 4\sqrt{3}F.$$

**Problem 50.** (Tsintsifas) Let  $p, q, r$  be positive real numbers and let  $a, b, c$  denote the sides of a triangle with area  $F$ . Then, we have

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq 2\sqrt{3}F.$$

**Problem 51.** (The Neuberg-Pedoe Inequality) Let  $a_1, b_1, c_1$  denote the sides of the triangle  $A_1B_1C_1$  with area  $F_1$ . Let  $a_2, b_2, c_2$  denote the sides of the triangle  $A_2B_2C_2$  with area  $F_2$ . Then, we have

$$a_1^2(b_2^2 + c_2^2 - a_2^2) + b_1^2(c_2^2 + a_2^2 - b_2^2) + c_1^2(a_2^2 + b_2^2 - c_2^2) \geq 16F_1F_2.$$

**Problem 52.** Let  $x_1, \dots, x_n$  be arbitrary real numbers. Prove the inequality.

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \dots + \frac{x_n}{1+x_1^2+\dots+x_n^2} < \sqrt{n}.$$

**Problem 53.** Let  $x, y$ , and  $z$  be positive numbers such that  $xyz \geq 1$ . Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

**Problem 54.** Prove that, for all  $x, y, z > 1$  such that  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ ,

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

**Problem 55.** Let  $a, b, c$  be positive numbers such that  $abc = 1$ . Prove that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

**Problem 56.** (Muirhead's Theorem) Let  $a_1, a_2, a_3, b_1, b_2, b_3$  be real numbers such that

$$a_1 \geq a_2 \geq a_3 \geq 0, b_1 \geq b_2 \geq b_3 \geq 0, a_1 \geq b_1, a_1 + a_2 \geq b_1 + b_2, a_1 + a_2 + a_3 = b_1 + b_2 + b_3.$$

Let  $x, y, z$  be positive real numbers. Then, we have

$$\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} \geq \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.$$

**Problem 57.** If  $m_a, m_b, m_c$  are medians and  $r_a, r_b, r_c$  the exradii of a triangle, prove that

$$\frac{r_a r_b}{m_a m_b} + \frac{r_b r_c}{m_b m_c} + \frac{r_c r_a}{m_c m_a} \geq 3.$$

**Problem 58.** Prove that, for all  $a, b, c > 0$ ,

$$\sqrt{a^4 + a^2 b^2 + b^4} + \sqrt{b^4 + b^2 c^2 + c^4} + \sqrt{c^4 + c^2 a^2 + a^4} \geq a\sqrt{2a^2 + bc} + b\sqrt{2b^2 + ca} + c\sqrt{2c^2 + ab}.$$

**Problem 59.** (Hölder's Inequality) Let  $x_{ij}$  ( $i = 1, \dots, m, j = 1, \dots, n$ ) be positive real numbers. Suppose that  $\omega_1, \dots, \omega_n$  are positive real numbers satisfying  $\omega_1 + \dots + \omega_n = 1$ . Then, we have

$$\prod_{j=1}^n \left( \sum_{i=1}^m x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^m \left( \prod_{j=1}^n x_{ij}^{\omega_j} \right).$$

**Problem 60.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a **continuous** function. Then, the followings are equivalent.

(1) For all  $n \in \mathbb{N}$ , the following inequality holds.

$$\omega_1 f(x_1) + \dots + \omega_n f(x_n) \geq f(\omega_1 x_1 + \dots + \omega_n x_n)$$

for all  $x_1, \dots, x_n \in [a, b]$  and  $\omega_1, \dots, \omega_n > 0$  with  $\omega_1 + \dots + \omega_n = 1$ .

(2) For all  $n \in \mathbb{N}$ , the following inequality holds.

$$r_1 f(x_1) + \dots + r_n f(x_n) \geq f(r_1 x_1 + \dots + r_n x_n)$$

for all  $x_1, \dots, x_n \in [a, b]$  and  $r_1, \dots, r_n \in \mathbb{Q}^+$  with  $r_1 + \dots + r_n = 1$ .

(3) For all  $N \in \mathbb{N}$ , the following inequality holds.

$$\frac{f(y_1) + \dots + f(y_N)}{N} \geq f\left(\frac{y_1 + \dots + y_N}{N}\right)$$

for all  $y_1, \dots, y_N \in [a, b]$ .

(4) For all  $k \in \{0, 1, 2, \dots\}$ , the following inequality holds.

$$\frac{f(y_1) + \dots + f(y_{2^k})}{2^k} \geq f\left(\frac{y_1 + \dots + y_{2^k}}{2^k}\right)$$

for all  $y_1, \dots, y_{2^k} \in [a, b]$ .

(5) We have  $\frac{1}{2}f(x) + \frac{1}{2}f(y) \geq f\left(\frac{x+y}{2}\right)$  for all  $x, y \in [a, b]$ .

(6) We have  $\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y)$  for all  $x, y \in [a, b]$  and  $\lambda \in (0, 1)$ .

### 3.3. Solutions.

**Solution 31.** The numerator of the difference  $\frac{1}{2} - \frac{u}{v}$  is equal to

$$\begin{aligned} v - 2u &= 2(ab + bc + ca) - (a^2 + b^2 + c^2) \\ &= a(b + c - a) + b(c + a - b) + c(a + b - c) \end{aligned}$$

By the triangle inequality,  $a < b + c$ ,  $b < c + a$  and  $c < a + b$ , so that the numerator is always positive.

Since all variables are positive, the right inequality follows.

Now the numerator of  $\frac{u}{v} - \frac{1}{3}$  is equal to

$$\begin{aligned} 3u - v &= 2(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= (a - b)^2 + (b - c)^2 + (c - a)^2 \end{aligned}$$

The right side, being a sum of squares, is nonnegative and it vanishes if and only if  $a = b = c$ . Thus the left inequality (and the equality) follows.

For the second part, i.e., to show that  $\frac{u}{v}$  can be arbitrarily close to  $\frac{1}{2}$ , let  $(a; b; c) = (\varepsilon; 1; 1)$  where  $0 < \varepsilon < 4$ . Then

$$\frac{1}{2} - \frac{u}{v} = \frac{\varepsilon(4 - \varepsilon)}{(2 + \varepsilon)^2}$$

This can be made as close to 0 as desired by taking  $\varepsilon$  sufficiently close to 0.

**Solution 32.** Since  $a$  and  $b$  are positive, the inequality is equivalent to

$$\left(\frac{a+b}{3}\right)^3 \geq \frac{a^2b}{4}$$

To prove this we can apply the arithmetic mean-geometric mean inequality to  $a/2$ ,  $a/2$ ,  $b$ . This gives

$$\frac{\frac{a}{2} + \frac{a}{2} + b}{3} \geq \sqrt[3]{\frac{a}{2} \frac{a}{2} b} = \sqrt[3]{\frac{a^2b}{4}}$$

Cubing both sides we get the required result and the equality holds in the AM-GM inequality when the averaged quantities are all equal, i.e., equality holds when  $b = a/2$ .

**Solution 33.** The given inequality is equivalent to

$$\frac{1}{1-2y} + \frac{2}{y} \geq \frac{25}{1+48(1-2y)y^2}, 0 < y \leq \frac{1}{2},$$

which implies

$$(1+48(1-2y)y^2)(2-3y) \geq 25y(1-2y).$$

Now,

$$\begin{aligned} & (1+48(1-2y)y^2)(2-3y) - 25y(1-2y) \\ &= (1+48y^2-96y^3)(2-3y) - 25y+50y^2 \\ &= 2-28y+146y^2-336y^3+288y^4 \\ &= 2(1-14y+73y^2-168y^3+144y^4) \\ &= 2(1-7y+12y^2)^2 \\ &= 2(3y-1)^2(4y-1)^2 \\ &\geq 0, \forall \text{ positive real } y \leq \frac{1}{2}, \end{aligned}$$

and the equality holds if  $y = \frac{1}{3}$  or  $\frac{1}{4}$ .

Hence proved.

**Solution 34.** Let each of  $b_1, b_2, \dots, b_n$  is equal to one of  $a_1, a_2, \dots, a_n$  such that  $b_1 < b_2 < \dots < b_n$ .

Then  $b_i \geq i$  for  $1 \leq i \leq n$ .

$$\begin{aligned} \text{Thus } \frac{a_1}{1^2} + \frac{a_2}{2^2} + \dots + \frac{a_n}{n^2} &\geq \frac{b_1}{1^2} + \frac{b_2}{2^2} + \dots + \frac{b_n}{n^2} \\ &\geq \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} \\ &\geq \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}. \end{aligned}$$

$$\begin{aligned} \text{Solution 35. } & \frac{x^5 - x^2}{x^5 + y^2 + z^2} - \frac{x^5 - x^2}{x^3(x^2 + y^2 + z^2)} = \frac{(x^3 - 1)^2 x^2 (y^2 + z^2)}{x^3(x^2 + y^2 + z^2)(x^5 + y^2 + z^2)} \geq \\ & 0 \\ & \frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq \frac{x^5 - x^2}{x^3(x^2 + y^2 + z^2)} + \frac{y^5 - y^2}{y^3(x^2 + y^2 + z^2)} + \\ & \frac{z^5 - z^2}{z^3(x^2 + y^2 + z^2)} \\ & \geq \frac{1}{x^2 + y^2 + z^2} \left( \left(x^2 - \frac{1}{x}\right) + \left(y^2 - \frac{1}{y}\right) + \left(z^2 - \frac{1}{z}\right) \right) \\ & \geq \frac{1}{x^2 + y^2 + z^2} [(x^2 - yz) + (y^2 - zx) + (z^2 - xy)] \\ & \geq 0 \end{aligned}$$

**Solution 36.** *The general term on the left hand side is:*

$$\frac{k}{10k+1} < \frac{k}{10k} = \frac{1}{10}$$

*Thus adding  $n$  terms on the left hand side we will get the inequality.*

**Solution 37.** *By the arithmetic-geometric means inequality, we have that*

$$\frac{2a^5 + 3u^5}{5} \geq \sqrt[5]{(a^5)^2(u^5)^3} = a^2u^3$$

*and similarly,*

$$\frac{2b^5 + 3v^5}{5} \geq b^2v^3$$

*Adding these two inequalities yields the result.*

**Solution 38.** *Here the expression is big but it is a very easy question. Please solve it.*

**Solution 39.**  $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n-1}} = \sqrt{n} - \sqrt{n-1}$

*Therefore  $\sqrt{n-1} - \sqrt{n+1} < 2\sqrt{n}$ .*

*Thus*

$$\sqrt{n-1} + \sqrt{n} + \sqrt{n+1} < 3\sqrt{n}.$$

*Squaring both sides we will get the required inequality.*

**Solution 40.** *The result is trivial for  $n = 2$ . Now for  $n=3$ , when  $x < y < z$ ,*

$$(xy^4 + yz^4 + zx^4) - (yx^4 + zy^4 + xz^4) = \frac{1}{2}(z-x)(y-x)(z-y)[(x+y)^2 + (x+z)^2 + (y+z)^2] \geq 0$$

*Now let us assume that the result is true for  $n \geq 3$ , then*

$$(a_1a_2^4 + a_2a_3^4 + \cdots + a_na_{n+1}^4 + a_{n+1}a_1^4) - (a_2a_1^4 + a_3a_2^4 + \cdots + a_{n+1}a_n^4 + a_1a_{n+1}^4)$$

$$= (a_1a_2^4 + a_2a_3^4 + \cdots + a_na_1^4) - (a_2a_1^4 + a_3a_2^4 + \cdots + a_1a_n^4) \\ + (a_1a_n^4 + a_na_{n+1}^4 + a_{n+1}a_1^4) - (a_na_1^4 + a_{n+1}a_n^4 + a_1a_{n+1}^4) \geq 0$$

*Thus proved.*

**Solution 41.** *We first notice that the constraint can be written as*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

*It is now enough to establish the auxiliary inequality*

$$\frac{x^4 + y^4}{xy(x^3 + y^3)} \geq \frac{1}{2} \left( \frac{1}{x} + \frac{1}{y} \right)$$

or

$$2(x^4 + y^4) \geq (x^3 + y^3)(x + y),$$

where  $x, y > 0$ . However, we obtain

$$2(x^4 + y^4) - (x^3 + y^3)(x + y) = x^4 + y^4 - x^3y - xy^3 = (x^3 - y^3)(x - y) \geq 0.$$

**Solution 42.** The left hand side admits the following decomposition

$$\frac{\sqrt{c+a-b} + \sqrt{a+b-c}}{2} + \frac{\sqrt{a+b-c} + \sqrt{b+c-a}}{2} + \frac{\sqrt{b+c-a} + \sqrt{c+a-b}}{2}.$$

We now use the inequality  $\frac{\sqrt{x} + \sqrt{y}}{2} \leq \sqrt{\frac{x+y}{2}}$  to deduce

$$\frac{\sqrt{c+a-b} + \sqrt{a+b-c}}{2} \leq \sqrt{a},$$

$$\frac{\sqrt{a+b-c} + \sqrt{b+c-a}}{2} \leq \sqrt{b},$$

$$\frac{\sqrt{b+c-a} + \sqrt{c+a-b}}{2} \leq \sqrt{c}.$$

Adding these three inequalities, we get the result.

**Solution 43.** Since the inequality is symmetric in the three variables, we may assume that  $a \geq b \geq c$ . We claim that

$$\frac{\sqrt{a+b-c}}{\sqrt{a} + \sqrt{b} - \sqrt{c}} \leq 1$$

and

$$\frac{\sqrt{b+c-a}}{\sqrt{b} + \sqrt{c} - \sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c} + \sqrt{a} - \sqrt{b}} \leq 2.$$

It is clear that the denominators are positive. So, the first inequality is equivalent to

$$\sqrt{a} + \sqrt{b} \geq \sqrt{a+b-c} + \sqrt{c}.$$

or

$$(\sqrt{a} + \sqrt{b})^2 \geq (\sqrt{a+b-c} + \sqrt{c})^2$$

or

$$\sqrt{ab} \geq \sqrt{c(a+b-c)}$$

or

$$ab \geq c(a+b-c),$$

which immediately follows from  $(a-c)(b-c) \geq 0$ . Now, we prove the second inequality. Setting  $p = \sqrt{a} + \sqrt{b}$  and  $q = \sqrt{a} - \sqrt{b}$ , we obtain  $a - b = pq$  and  $p \geq 2\sqrt{c}$ . It now becomes

$$\frac{\sqrt{c-pq}}{\sqrt{c}-q} + \frac{\sqrt{c+pq}}{\sqrt{c}+q} \leq 2.$$



We now apply The Cauchy-Schwartz Inequality to deduce

$$\begin{aligned}
\left(\frac{\sqrt{c-pq}}{\sqrt{c-q}} + \frac{\sqrt{c+pq}}{\sqrt{c+q}}\right)^2 &\leq \left(\frac{c-pq}{\sqrt{c-q}} + \frac{c+pq}{\sqrt{c+q}}\right) \left(\frac{1}{\sqrt{c-q}} + \frac{1}{\sqrt{c+q}}\right) \\
&= \frac{2(c\sqrt{c}-pq^2)}{c-q^2} \cdot \frac{2\sqrt{c}}{c-q^2} \\
&= 4 \frac{c^2 - \sqrt{c}pq^2}{(c-q^2)^2} \\
&\leq 4 \frac{c^2 - 2cq^2}{(c-q^2)^2} \\
&\leq 4 \frac{c^2 - 2cq^2 + q^4}{(c-q^2)^2} \\
&\leq 4.
\end{aligned}$$

We find that the equality holds if and only if  $a = b = c$ .

**Solution 44.** We observe that the sine function is not concave on  $[0, 3\pi]$  and that it is negative on  $(\pi, 2\pi)$ . Since the inequality is symmetric in the three variables, we may assume that  $A \leq B \leq C$ . Observe that  $A+B+C = \pi$  and that  $3A, 3B, 3C \in [0, 3\pi]$ . It is clear that  $A \leq \frac{\pi}{3} \leq C$ .

We see that either  $3B \in [2\pi, 3\pi)$  or  $3C \in (0, \pi)$  is impossible. In the case when  $3B \in [\pi, 2\pi)$ , we obtain the estimation

$$\sin 3A + \sin 3B + \sin 3C \leq 1 + 0 + 1 = 2 < \frac{3\sqrt{3}}{2}.$$

So, we may assume that  $3B \in (0, \pi)$ . Similarly, in the case when  $3C \in [\pi, 2\pi]$ , we obtain

$$\sin 3A + \sin 3B + \sin 3C \leq 1 + 1 + 0 = 2 < \frac{3\sqrt{3}}{2}.$$

Hence, we also assume  $3C \in (2\pi, 3\pi)$ . Now, our assumptions become  $A \leq B < \frac{1}{3}\pi$  and  $\frac{2}{3}\pi < C$ . After the substitution  $\theta = C - \frac{2}{3}\pi$ , the trigonometric inequality becomes

$$\sin 3A + \sin 3B + \sin 3\theta \leq \frac{3\sqrt{3}}{2}.$$

Since  $3A, 3B, 3\theta \in (0, \pi)$  and since the sine function is concave on  $[0, \pi]$ , Jensen's Inequality gives

$$\sin 3A + \sin 3B + \sin 3\theta \leq 3 \sin \left( \frac{3A + 3B + 3\theta}{3} \right) = 3 \sin \left( \frac{3A + 3B + 3C - 2\pi}{3} \right) = 3 \sin \left( \frac{\pi}{3} \right).$$

Under the assumption  $A \leq B \leq C$ , the equality occurs only when  $(A, B, C) = (\frac{1}{9}\pi, \frac{1}{9}\pi, \frac{7}{9}\pi)$ .

**Solution 45.** We observe that two sequences are similarly ordered in the sense that

$$(x_i - x_j)(y_i - y_j) \geq 0$$

for all  $1 \leq i, j \leq n$ . Now, the given inequality is an immediate consequence of the identity

$$\frac{1}{n} \sum_{i=1}^n x_i y_i - \frac{1}{n} \left( \sum_{i=1}^n x_i \right) \frac{1}{n} \left( \sum_{i=1}^n y_i \right) = \frac{1}{n^2} \sum_{1 \leq i, j \leq n} (x_i - x_j)(y_i - y_j).$$

**Solution 46.** The key observation is the following identity:

$$\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} = \frac{1}{2} \sum_{i=1}^n \frac{a_i^2 + b_i^2}{a_i + b_i},$$

which is equivalent to

$$\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} = \sum_{i=1}^n \frac{b_i^2}{a_i + b_i},$$

which immediately follows from

$$\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} - \sum_{i=1}^n \frac{b_i^2}{a_i + b_i} = \sum_{i=1}^n \frac{a_i^2 - b_i^2}{a_i + b_i} = \sum_{i=1}^n (a_i - b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i = 0.$$

Our strategy is to establish the following symmetric inequality

$$\frac{1}{2} \sum_{i=1}^n \frac{a_i^2 + b_i^2}{a_i + b_i} \geq \frac{a_1 + \cdots + a_n + b_1 + \cdots + b_n}{4}.$$

It now remains to check the the auxiliary inequality

$$\frac{a^2 + b^2}{a + b} \geq \frac{a + b}{2},$$

where  $a, b > 0$ . Indeed, we have  $2(a^2 + b^2) - (a + b)^2 = (a - b)^2 \geq 0$ .

**Solution 47.** Since the constraint  $ab + bc + cd + da = 1$  is not symmetric in the variables, we cannot consider the case when  $a \geq b \geq c \geq d$  only. We first make the observation that

$$a^2 + b^2 + c^2 + d^2 = \frac{a^2 + b^2}{2} + \frac{b^2 + c^2}{2} + \frac{c^2 + d^2}{2} + \frac{d^2 + a^2}{2} \geq ab + bc + cd + da = 1.$$

Our strategy is to establish the following result. It is symmetric.

Let  $a, b, c, d \geq 0$  with  $a^2 + b^2 + c^2 + d^2 \geq 1$ . Then, we obtain

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

We now exploit the symmetry! Since everything is symmetric in the variables, we may assume that  $a \geq b \geq c \geq d$ . Two applications of Chebyshev's Inequality and one application of The AM-GM Inequality yield

$$\begin{aligned} & \frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \\ & \geq \frac{1}{4} (a^3 + b^3 + c^3 + d^3) \left( \frac{1}{b+c+d} + \frac{1}{c+d+a} + \frac{1}{d+a+b} + \frac{1}{a+b+c} \right) \\ & \geq \frac{1}{4} (a^3 + b^3 + c^3 + d^3) \frac{4^2}{(b+c+d) + (c+d+a) + (d+a+b) + (a+b+c)} \\ & \geq \frac{1}{4^2} (a^2 + b^2 + c^2 + d^2) (a+b+c+d) \frac{4^2}{3(a+b+c+d)} \\ & = \frac{1}{3}. \end{aligned}$$

**Solution 48.** Write  $a = y + z$ ,  $b = z + x$ ,  $c = x + y$  for  $x, y, z > 0$ . It's equivalent to

$$((y+z)^2 + (z+x)^2 + (x+y)^2)^2 \geq 48(x+y+z)xyz,$$

which can be obtained as following :

$$((y+z)^2 + (z+x)^2 + (x+y)^2)^2 \geq 16(yz+zx+xy)^2 \geq 16 \cdot 3(xy \cdot yz + yz \cdot zx + xy \cdot yz).$$

Here, we used the well-known inequalities  $p^2 + q^2 \geq 2pq$  and  $(p+q+r)^2 \geq 3(pq + qr + rp)$ .

**Solution 49.** After the substitution  $a = y + z$ ,  $b = z + x$ ,  $c = x + y$ , where  $x, y, z > 0$ , it becomes

$$xy + yz + zx \geq \sqrt{3xyz(x+y+z)},$$

which follows from the identity

$$(xy+yz+zx)^2 - 3xyz(x+y+z) = \frac{(xy - yz)^2 + (yz - zx)^2 + (zx - xy)^2}{2}.$$

**Solution 50.** By Hadwiger-Finsler Inequality, it suffices to show that

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq \frac{1}{2}(a+b+c)^2 - (a^2 + b^2 + c^2)$$

or

$$\left(\frac{p+q+r}{q+r}\right)a^2 + \left(\frac{p+q+r}{r+p}\right)b^2 + \left(\frac{p+q+r}{p+q}\right)c^2 \geq \frac{1}{2}(a+b+c)^2$$

or

$$((q+r) + (r+p) + (p+q)) \left(\frac{1}{q+r}a^2 + \frac{1}{r+p}b^2 + \frac{1}{p+q}c^2\right) \geq (a+b+c)^2.$$

However, this is a straightforward consequence of The Cauchy-Schwarz Inequality.

**Solution 51.** We begin with the following lemma.

**Lemma 3.1.** We have

$$a_1^2(a_2^2 + b_2^2 - c_2^2) + b_1^2(b_2^2 + c_2^2 - a_2^2) + c_1^2(c_2^2 + a_2^2 - b_2^2) > 0.$$

*Proof.* Observe that it's equivalent to

$$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) > 2(a_1^2a_2^2 + b_1^2b_2^2 + c_1^2c_2^2).$$

From Heron's Formula, we find that, for  $i = 1, 2$ ,

$$16F_i^2 = (a_i^2 + b_i^2 + c_i^2)^2 - 2(a_i^4 + b_i^4 + c_i^4) > 0 \quad \text{or} \quad a_i^2 + b_i^2 + c_i^2 > \sqrt{2(a_i^4 + b_i^4 + c_i^4)}.$$

The Cauchy-Schwarz Inequality implies that

$$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) > 2\sqrt{(a_1^4 + b_1^4 + c_1^4)(a_2^4 + b_2^4 + c_2^4)} \geq 2(a_1^2a_2^2 + b_1^2b_2^2 + c_1^2c_2^2).$$

□

By the lemma, we obtain

$$L = a_1^2(b_2^2 + c_2^2 - a_2^2) + b_1^2(c_2^2 + a_2^2 - b_2^2) + c_1^2(a_2^2 + b_2^2 - c_2^2) > 0,$$

Hence, we need to show that

$$L^2 - (16F_1^2)(16F_2^2) \geq 0.$$

One may easily check the following identity

$$L^2 - (16F_1^2)(16F_2^2) = -4(UV + VW + WU),$$

where

$$U = b_1^2c_2^2 - b_2^2c_1^2, \quad V = c_1^2a_2^2 - c_2^2a_1^2 \quad \text{and} \quad W = a_1^2b_2^2 - a_2^2b_1^2.$$

Using the identity

$$a_1^2U + b_1^2V + c_1^2W = 0 \quad \text{or} \quad W = -\frac{a_1^2}{c_1^2}U - \frac{b_1^2}{c_1^2}V,$$

one may also deduce that

$$UV + VW + WU = -\frac{a_1^2}{c_1^2} \left( U - \frac{c_1^2 - a_1^2 - b_1^2}{2a_1^2} V \right)^2 - \frac{4a_1^2b_1^2 - (c_1^2 - a_1^2 - b_1^2)^2}{4a_1^2c_1^2} V^2.$$

It follows that

$$UV + VW + WU = -\frac{a_1^2}{c_1^2} \left( U - \frac{c_1^2 - a_1^2 - b_1^2}{2a_1^2} V \right)^2 - \frac{16F_1^2}{4a_1^2 c_1^2} V^2 \leq 0.$$

**Solution 52.** We only consider the case when  $x_1, \dots, x_n$  are all non-negative real numbers. (Why?)<sup>1</sup> Let  $x_0 = 1$ . After the substitution  $y_i = x_0^2 + \dots + x_i^2$  for all  $i = 0, \dots, n$ , we obtain  $x_i = \sqrt{y_i - y_{i-1}}$ . We need to prove the following inequality

$$\sum_{i=0}^n \frac{\sqrt{y_i - y_{i-1}}}{y_i} < \sqrt{n}.$$

Since  $y_i \geq y_{i-1}$  for all  $i = 1, \dots, n$ , we have an upper bound of the left hand side:

$$\sum_{i=0}^n \frac{\sqrt{y_i - y_{i-1}}}{y_i} \leq \sum_{i=0}^n \frac{\sqrt{y_i - y_{i-1}}}{\sqrt{y_i y_{i-1}}} = \sum_{i=0}^n \sqrt{\frac{1}{y_{i-1}} - \frac{1}{y_i}}$$

We now apply the Cauchy-Schwarz inequality to give an upper bound of the last term:

$$\sum_{i=0}^n \sqrt{\frac{1}{y_{i-1}} - \frac{1}{y_i}} \leq \sqrt{n \sum_{i=0}^n \left( \frac{1}{y_{i-1}} - \frac{1}{y_i} \right)} = \sqrt{n \left( \frac{1}{y_0} - \frac{1}{y_n} \right)}.$$

Since  $y_0 = 1$  and  $y_n > 0$ , this yields the desired upper bound  $\sqrt{n}$ .

**Solution 53.** It's equivalent to the following inequality

$$\left( \frac{x^2 - x^5}{x^5 + y^2 + z^2} + 1 \right) + \left( \frac{y^2 - y^5}{y^5 + z^2 + x^2} + 1 \right) + \left( \frac{z^2 - z^5}{z^5 + x^2 + y^2} + 1 \right) \leq 3$$

or

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \leq 3.$$

With The Cauchy-Schwarz Inequality and the fact that  $xyz \geq 1$ , we have

$$(x^5 + y^2 + z^2)(yz + y^2 + z^2) \geq (x^2 + y^2 + z^2)^2$$

or

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} \leq \frac{yz + y^2 + z^2}{x^2 + y^2 + z^2}.$$

Taking the cyclic sum, we reach

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \leq 2 + \frac{xy + yz + zx}{x^2 + y^2 + z^2} \leq 3.$$

---

<sup>1</sup>  $\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \dots + \frac{x_n}{1+x_1^2+\dots+x_n^2} \leq \frac{|x_1|}{1+x_1^2} + \frac{|x_2|}{1+x_1^2+x_2^2} + \dots + \frac{|x_n|}{1+x_1^2+\dots+x_n^2}.$

**Solution 54.** After the algebraic substitution  $a = \frac{1}{x}$ ,  $b = \frac{1}{y}$ ,  $c = \frac{1}{z}$ , we are required to prove that

$$\sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \geq \sqrt{\frac{1-a}{a}} + \sqrt{\frac{1-b}{b}} + \sqrt{\frac{1-c}{c}},$$

where  $a, b, c \in (0, 1)$  and  $a+b+c = 2$ . Using the constraint  $a+b+c = 2$ , we obtain a homogeneous inequality

$$\sqrt{\frac{1}{2}(a+b+c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)} \geq \sqrt{\frac{\frac{a+b+c}{2} - a}{a}} + \sqrt{\frac{\frac{a+b+c}{2} - b}{b}} + \sqrt{\frac{\frac{a+b+c}{2} - c}{c}}$$

or

$$\sqrt{(a+b+c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)} \geq \sqrt{\frac{b+c-a}{a}} + \sqrt{\frac{c+a-b}{b}} + \sqrt{\frac{a+b-c}{c}},$$

which immediately follows from The Cauchy-Schwarz Inequality

$$\sqrt{[(b+c-a) + (c+a-b) + (a+b-c)] \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)} \geq \sqrt{\frac{b+c-a}{a}} + \sqrt{\frac{c+a-b}{b}} + \sqrt{\frac{a+b-c}{c}}$$

**Solution 55.** We can rewrite the given inequality as following :

$$\frac{1}{a+b+(abc)^{1/3}} + \frac{1}{b+c+(abc)^{1/3}} + \frac{1}{c+a+(abc)^{1/3}} \leq \frac{1}{(abc)^{1/3}}.$$

We make the substitution  $a = x^3, b = y^3, c = z^3$  with  $x, y, z > 0$ . Then, it becomes

$$\frac{1}{x^3 + y^3 + xyz} + \frac{1}{y^3 + z^3 + xyz} + \frac{1}{z^3 + x^3 + xyz} \leq \frac{1}{xyz}$$

which is equivalent to

$$xyz \sum_{\text{cyclic}} (x^3 + y^3 + xyz)(y^3 + z^3 + xyz) \leq (x^3 + y^3 + xyz)(y^3 + z^3 + xyz)(z^3 + x^3 + xyz)$$

or

$$\sum_{\text{sym}} x^6 y^3 \geq \sum_{\text{sym}} x^5 y^2 z^2 \quad !$$

We now obtain

$$\begin{aligned}
\sum_{\text{sym}} x^6 y^3 &= \sum_{\text{cyclic}} x^6 y^3 + y^6 x^3 \\
&\geq \sum_{\text{cyclic}} x^5 y^4 + y^5 x^4 \\
&= \sum_{\text{cyclic}} x^5 (y^4 + z^4) \\
&\geq \sum_{\text{cyclic}} x^5 (y^2 z^2 + y^2 z^2) \\
&= \sum_{\text{sym}} x^5 y^2 z^2.
\end{aligned}$$

**Solution 56.** We distinguish two cases.

*Case 1.*  $b_1 \geq a_2$ : It follows from  $a_1 \geq a_1 + a_2 - b_1$  and from  $a_1 \geq b_1$  that  $a_1 \geq \max(a_1 + a_2 - b_1, b_1)$  so that  $\max(a_1, a_2) = a_1 \geq \max(a_1 + a_2 - b_1, b_1)$ . From  $a_1 + a_2 - b_1 \geq b_1 + a_3 - b_1 = a_3$  and  $a_1 + a_2 - b_1 \geq b_2 \geq b_3$ , we have  $\max(a_1 + a_2 - b_1, a_3) \geq \max(b_2, b_3)$ . It follows that

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} &= \sum_{\text{cyclic}} z^{a_3} (x^{a_1} y^{a_2} + x^{a_2} y^{a_1}) \\
&\geq \sum_{\text{cyclic}} z^{a_3} (x^{a_1 + a_2 - b_1} y^{b_1} + x^{b_1} y^{a_1 + a_2 - b_1}) \\
&= \sum_{\text{cyclic}} x^{b_1} (y^{a_1 + a_2 - b_1} z^{a_3} + y^{a_3} z^{a_1 + a_2 - b_1}) \\
&\geq \sum_{\text{cyclic}} x^{b_1} (y^{b_2} z^{b_3} + y^{b_3} z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.
\end{aligned}$$

*Case 2.*  $b_1 \leq a_2$ : It follows from  $3b_1 \geq b_1 + b_2 + b_3 = a_1 + a_2 + a_3 \geq b_1 + a_2 + a_3$  that  $b_1 \geq a_2 + a_3 - b_1$  and that  $a_1 \geq a_2 \geq b_1 \geq a_2 + a_3 - b_1$ . Therefore, we have  $\max(a_2, a_3) \geq \max(b_1, a_2 + a_3 - b_1)$  and

$\max(a_1, a_2 + a_3 - b_1) \geq \max(b_2, b_3)$ . It follows that

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} &= \sum_{\text{cyclic}} x^{a_1} (y^{a_2} z^{a_3} + y^{a_3} z^{a_2}) \\
&\geq \sum_{\text{cyclic}} x^{a_1} (y^{b_1} z^{a_2+a_3-b_1} + y^{a_2+a_3-b_1} z^{b_1}) \\
&= \sum_{\text{cyclic}} y^{b_1} (x^{a_1} z^{a_2+a_3-b_1} + x^{a_2+a_3-b_1} z^{a_1}) \\
&\geq \sum_{\text{cyclic}} y^{b_1} (x^{b_2} z^{b_3} + x^{b_3} z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.
\end{aligned}$$

**Solution 57.** Set  $2s = a + b + c$ . Using the well-known identities

$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}, \quad m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}, \quad \text{etc.}$$

we obtain

$$\sum_{\text{cyclic}} \frac{r_b r_c}{m_b m_c} = \sum_{\text{cyclic}} \frac{4s(s-a)}{\sqrt{(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)}}.$$

Applying the AM-GM inequality, we obtain

$$\sum_{\text{cyclic}} \frac{r_b r_c}{m_b m_c} \geq \sum_{\text{cyclic}} \frac{8s(s-a)}{(2c^2 + 2a^2 - b^2) + (2a^2 + 2b^2 - c^2)} = \sum_{\text{cyclic}} \frac{2(a+b+c)(b+c-a)}{4a^2 + b^2 + c^2}.$$

Thus, it will be enough to show that

$$\sum_{\text{cyclic}} \frac{2(a+b+c)(b+c-a)}{4a^2 + b^2 + c^2} \geq 3.$$

After expanding the above inequality, we see that it becomes

$$2 \sum_{\text{cyclic}} a^6 + 4 \sum_{\text{cyclic}} a^4 bc + 20 \sum_{\text{sym}} a^3 b^2 c + 68 \sum_{\text{cyclic}} a^3 b^3 + 16 \sum_{\text{cyclic}} a^5 b \geq 276 a^2 b^2 c^2 + 27 \sum_{\text{cyclic}} a^4 b^2.$$

We note that this cannot be proven by just applying Muirhead's Theorem. Since  $a, b, c$  are the sides of a triangle, we can make The Ravi Substitution  $a = y + z$ ,  $b = z + x$ ,  $c = x + y$ , where  $x, y, z > 0$ . After some brute-force algebra, we can rewrite the above inequality as

$$\begin{aligned}
25 \sum_{\text{sym}} x^6 + 230 \sum_{\text{sym}} x^5 y + 115 \sum_{\text{sym}} x^4 y^2 + 10 \sum_{\text{sym}} x^3 y^3 + 80 \sum_{\text{sym}} x^4 y z \\
\geq 336 \sum_{\text{sym}} x^3 y^2 z + 124 \sum_{\text{sym}} x^2 y^2 z^2.
\end{aligned}$$



Now, by Muirhead's Theorem, we get the result !

**Solution 58.** We obtain the chain of equalities and inequalities

$$\begin{aligned}
\sum_{\text{cyclic}} \sqrt{a^4 + a^2b^2 + b^4} &= \sum_{\text{cyclic}} \sqrt{\left(a^4 + \frac{a^2b^2}{2}\right) + \left(b^4 + \frac{a^2b^2}{2}\right)} \\
&\geq \frac{1}{\sqrt{2}} \sum_{\text{cyclic}} \left( \sqrt{a^4 + \frac{a^2b^2}{2}} + \sqrt{b^4 + \frac{a^2b^2}{2}} \right) && \text{(Cauchy - Schwarz)} \\
&= \frac{1}{\sqrt{2}} \sum_{\text{cyclic}} \left( \sqrt{a^4 + \frac{a^2b^2}{2}} + \sqrt{a^4 + \frac{a^2c^2}{2}} \right) \\
&\geq \sqrt{2} \sum_{\text{cyclic}} \sqrt[4]{\left(a^4 + \frac{a^2b^2}{2}\right) \left(a^4 + \frac{a^2c^2}{2}\right)} && \text{(AM - GM)} \\
&\geq \sqrt{2} \sum_{\text{cyclic}} \sqrt{a^4 + \frac{a^2bc}{2}} && \text{(Cauchy - Schwarz)} \\
&= \sum_{\text{cyclic}} \sqrt{2a^4 + a^2bc}.
\end{aligned}$$

**Solution 59.** Because of the homogeneity of the inequality, we may rescale  $x_{1j}, \dots, x_{mj}$  so that  $x_{1j} + \dots + x_{mj} = 1$  for each  $j \in \{1, \dots, n\}$ . Then, we need to show that

$$\prod_{j=1}^n 1^{\omega_j} \geq \sum_{i=1}^m \prod_{j=1}^n x_{ij}^{\omega_j} \quad \text{or} \quad 1 \geq \sum_{i=1}^m \prod_{j=1}^n x_{ij}^{\omega_j}.$$

The Weighted AM-GM Inequality provides that

$$\sum_{j=1}^n \omega_j x_{ij} \geq \prod_{j=1}^n x_{ij}^{\omega_j} \quad (i \in \{1, \dots, m\}) \implies \sum_{i=1}^m \sum_{j=1}^n \omega_j x_{ij} \geq \sum_{i=1}^m \prod_{j=1}^n x_{ij}^{\omega_j}.$$

However, we immediately have

$$\sum_{i=1}^m \sum_{j=1}^n \omega_j x_{ij} = \sum_{j=1}^n \sum_{i=1}^m \omega_j x_{ij} = \sum_{j=1}^n \omega_j \left( \sum_{i=1}^m x_{ij} \right) = \sum_{j=1}^n \omega_j = 1.$$

**Solution 60.** (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4)  $\Rightarrow$  (5) is obvious.

(2)  $\Rightarrow$  (1) : Let  $x_1, \dots, x_n \in [a, b]$  and  $\omega_1, \dots, \omega_n > 0$  with  $\omega_1 + \dots + \omega_n = 1$ . One may see that there exist positive rational sequences  $\{r_k(1)\}_{k \in \mathbb{N}}, \dots, \{r_k(n)\}_{k \in \mathbb{N}}$  satisfying

$$\lim_{k \rightarrow \infty} r_k(j) = \omega_j \quad (1 \leq j \leq n) \quad \text{and} \quad r_k(1) + \dots + r_k(n) = 1 \quad \text{for all } k \in \mathbb{N}.$$

By the hypothesis in (2), we obtain  $r_k(1)f(x_1) + \cdots + r_k(n)f(x_n) \geq f(r_k(1)x_1 + \cdots + r_k(n)x_n)$ . Since  $f$  is continuous, taking  $k \rightarrow \infty$  to both sides yields the inequality

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1 x_1 + \cdots + \omega_n x_n).$$

(3)  $\Rightarrow$  (2) : Let  $x_1, \dots, x_n \in [a, b]$  and  $r_1, \dots, r_n \in \mathbb{Q}^+$  with  $r_1 + \cdots + r_n = 1$ . We can find a positive integer  $N \in \mathbb{N}$  so that  $Nr_1, \dots, Nr_n \in \mathbb{N}$ . For each  $i \in \{1, \dots, n\}$ , we can write  $r_i = \frac{p_i}{N}$ , where  $p_i \in \mathbb{N}$ . It follows from  $r_1 + \cdots + r_n = 1$  that  $N = p_1 + \cdots + p_n$ . Then, (3) implies that

$$\begin{aligned} & r_1 f(x_1) + \cdots + r_n f(x_n) \\ &= \frac{\overbrace{f(x_1) + \cdots + f(x_1)}^{p_1 \text{ terms}} + \cdots + \overbrace{f(x_n) + \cdots + f(x_n)}^{p_n \text{ terms}}}{N} \\ &\geq f\left(\frac{\overbrace{x_1 + \cdots + x_1}^{p_1 \text{ terms}} + \cdots + \overbrace{x_n + \cdots + x_n}^{p_n \text{ terms}}}{N}\right) \\ &= f(r_1 x_1 + \cdots + r_n x_n). \end{aligned}$$

(4)  $\Rightarrow$  (3) : Let  $y_1, \dots, y_N \in [a, b]$ . Take a large  $k \in \mathbb{N}$  so that  $2^k > N$ . Let  $a = \frac{y_1 + \cdots + y_N}{N}$ . Then, (4) implies that

$$\begin{aligned} & \frac{f(y_1) + \cdots + f(y_N) + (2^k - n)f(a)}{2^k} \\ &= \frac{f(y_1) + \cdots + f(y_N) + \overbrace{f(a) + \cdots + f(a)}^{(2^k - N) \text{ terms}}}{2^k} \\ &\geq f\left(\frac{y_1 + \cdots + y_N + \overbrace{a + \cdots + a}^{(2^k - N) \text{ terms}}}{2^k}\right) \\ &= f(a) \end{aligned}$$

so that

$$f(y_1) + \cdots + f(y_N) \geq Nf(a) = Nf\left(\frac{y_1 + \cdots + y_N}{N}\right).$$

(5)  $\Rightarrow$  (4) : We use induction on  $k$ . In case  $k = 0, 1, 2$ , it clearly holds. Suppose that (4) holds for some  $k \geq 2$ . Let  $y_1, \dots, y_{2^{k+1}} \in [a, b]$ . By

the induction hypothesis, we obtain

$$\begin{aligned}
& f(y_1) + \cdots + f(y_{2^k}) + f(y_{2^{k+1}}) + \cdots + f(y_{2^{k+1}}) \\
\geq & 2^k f\left(\frac{y_1 + \cdots + y_{2^k}}{2^k}\right) + 2^k f\left(\frac{y_{2^{k+1}} + \cdots + y_{2^{k+1}}}{2^k}\right) \\
= & 2^{k+1} \frac{f\left(\frac{y_1 + \cdots + y_{2^k}}{2^k}\right) + f\left(\frac{y_{2^{k+1}} + \cdots + y_{2^{k+1}}}{2^k}\right)}{2} \\
\geq & 2^{k+1} f\left(\frac{\frac{y_1 + \cdots + y_{2^k}}{2^k} + \frac{y_{2^{k+1}} + \cdots + y_{2^{k+1}}}{2^k}}{2}\right) \\
= & 2^{k+1} f\left(\frac{y_1 + \cdots + y_{2^{k+1}}}{2^{k+1}}\right).
\end{aligned}$$

Hence, (4) holds for  $k + 1$ . This completes the induction.

So far, we've established that (1), (2), (3), (4), (5) are all equivalent. Since (1)  $\Rightarrow$  (6)  $\Rightarrow$  (5) is obvious, this completes the proof.

## 4. PROBLEMS FOR PRACTICE

Try to prove the following inequalities.

## 4.1. General Problems.

**Practice Problem 1.** (BMO 2005, Proposed by Serbia and Montenegro) ( $a, b, c > 0$ )

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}$$

**Practice Problem 2.** (Romania 2005, Cezar Lupu) ( $a, b, c > 0$ )

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

**Practice Problem 3.** (Romania 2005, Traian Tamaian) ( $a, b, c > 0$ )

$$\frac{a}{b+2c+d} + \frac{b}{c+2d+a} + \frac{c}{d+2a+b} + \frac{d}{a+2b+c} \geq 1$$

**Practice Problem 4.** (Romania 2005, Cezar Lupu) ( $a+b+c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ ,  $a, b, c > 0$ )

$$a+b+c \geq \frac{3}{abc}$$

**Practice Problem 5.** (Romania 2005, Cezar Lupu) ( $1 = (a+b)(b+c)(c+a)$ ,  $a, b, c > 0$ )

$$ab+bc+ca \geq \frac{3}{4}$$

**Practice Problem 6.** (Romania 2005, Robert Szasz) ( $a+b+c = 3$ ,  $a, b, c > 0$ )

$$a^2b^2c^2 \geq (3-2a)(3-2b)(3-2c)$$

**Practice Problem 7.** (Romania 2005) ( $abc \geq 1$ ,  $a, b, c > 0$ )

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq 1$$

**Practice Problem 8. (Romania 2005, Unused)** ( $abc = 1$ ,  $a, b, c > 0$ )

$$\frac{a}{b^2(c+1)} + \frac{b}{c^2(a+1)} + \frac{c}{a^2(b+1)} \geq \frac{3}{2}$$

**Practice Problem 9. (Romania 2005, Unused)** ( $a+b+c \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ ,  $a, b, c > 0$ )

$$\frac{a^3c}{b(c+a)} + \frac{b^3a}{c(a+b)} + \frac{c^3b}{a(b+c)} \geq \frac{3}{2}$$

**Practice Problem 10. (Romania 2005, Unused)** ( $a+b+c = 1$ ,  $a, b, c > 0$ )

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}$$

**Practice Problem 11. (Romania 2005, Unused)** ( $ab+bc+ca+2abc = 1$ ,  $a, b, c > 0$ )

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq \frac{3}{2}$$

**Practice Problem 12. (Chzech and Solvak 2005)** ( $abc = 1$ ,  $a, b, c > 0$ )

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}$$

**Practice Problem 13. (Japan 2005)** ( $a+b+c = 1$ ,  $a, b, c > 0$ )

$$a(1+b-c)^{\frac{1}{3}} + b(1+c-a)^{\frac{1}{3}} + c(1+a-b)^{\frac{1}{3}} \leq 1$$

**Practice Problem 14. (Germany 2005)** ( $a+b+c = 1$ ,  $a, b, c > 0$ )

$$2 \left( \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \right) \geq \frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+c}{1-c}$$

**Practice Problem 15. (Vietnam 2005)** ( $a, b, c > 0$ )

$$\left( \frac{a}{a+b} \right)^3 + \left( \frac{b}{b+c} \right)^3 + \left( \frac{c}{c+a} \right)^3 \geq \frac{3}{8}$$

**Practice Problem 16. (China 2005)** ( $a + b + c = 1$ ,  $a, b, c > 0$ )

$$10(a^3 + b^3 + c^3) - 9(a^5 + b^5 + c^5) \geq 1$$

**Practice Problem 17. (China 2005)** ( $abcd = 1$ ,  $a, b, c, d > 0$ )

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1$$

**Practice Problem 18. (China 2005)** ( $ab + bc + ca = \frac{1}{3}$ ,  $a, b, c \geq 0$ )

$$\frac{1}{a^2 - bc + 1} + \frac{1}{b^2 - ca + 1} + \frac{1}{c^2 - ab + 1} \leq 3$$

**Practice Problem 19. (Poland 2005)** ( $0 \leq a, b, c \leq 1$ )

$$\frac{a}{bc+1} + \frac{b}{ca+1} + \frac{c}{ab+1} \leq 2$$

**Practice Problem 20. (Poland 2005)** ( $ab + bc + ca = 3$ ,  $a, b, c > 0$ )

$$a^3 + b^3 + c^3 + 6abc \geq 9$$

**Practice Problem 21. (Baltic Way 2005)** ( $abc = 1$ ,  $a, b, c > 0$ )

$$\frac{a}{a^2+2} + \frac{b}{b^2+2} + \frac{c}{c^2+2} \geq 1$$

**Practice Problem 22. (Serbia and Montenegro 2005)** ( $a, b, c > 0$ )

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}(a+b+c)}$$

**Practice Problem 23. (Serbia and Montenegro 2005)** ( $a + b + c = 3$ ,  $a, b, c > 0$ )

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

**Practice Problem 24. (Bosnia and Hercegovina 2005)** ( $a + b + c = 1$ ,  $a, b, c > 0$ )

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \leq \frac{1}{\sqrt{3}}$$

**Practice Problem 25. (Iran 2005)** ( $a, b, c > 0$ )

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \geq (a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

**Practice Problem 26. (Austria 2005)** ( $a, b, c, d > 0$ )

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \geq \frac{a + b + c + d}{abcd}$$

**Practice Problem 27. (Moldova 2005)** ( $a^4 + b^4 + c^4 = 3$ ,  $a, b, c > 0$ )

$$\frac{1}{4 - ab} + \frac{1}{4 - bc} + \frac{1}{4 - ca} \leq 1$$

**Practice Problem 28. (APMO 2005)** ( $abc = 8$ ,  $a, b, c > 0$ )

$$\frac{a^2}{\sqrt{(1 + a^3)(1 + b^3)}} + \frac{b^2}{\sqrt{(1 + b^3)(1 + c^3)}} + \frac{c^2}{\sqrt{(1 + c^3)(1 + a^3)}} \geq \frac{4}{3}$$

**Practice Problem 29. (IMO 2005)** ( $xyz \geq 1$ ,  $x, y, z > 0$ )

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0$$

**Practice Problem 30. (Poland 2004)** ( $a + b + c = 0$ ,  $a, b, c \in \mathbb{R}$ )

$$b^2c^2 + c^2a^2 + a^2b^2 + 3 \geq 6abc$$

**Practice Problem 31. (Baltic Way 2004)** ( $abc = 1$ ,  $a, b, c > 0$ ,  $n \in \mathbb{N}$ )

$$\frac{1}{a^n + b^n + 1} + \frac{1}{b^n + c^n + 1} + \frac{1}{c^n + a^n + 1} \leq 1$$

**Practice Problem 32. (Junior Balkan 2004)** ( $(x, y) \in \mathbb{R}^2 - \{(0, 0)\}$ )

$$\frac{2\sqrt{2}}{x^2 + y^2} \geq \frac{x + y}{x^2 - xy + y^2}$$

**Practice Problem 33.** (IMO Short List 2004) ( $ab + bc + ca = 1$ ,  $a, b, c > 0$ )

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}$$

**Practice Problem 34.** (APMO 2004) ( $a, b, c > 0$ )

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca)$$

**Practice Problem 35.** (USA 2004) ( $a, b, c > 0$ )

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$$

**Practice Problem 36.** (Junior BMO 2003) ( $x, y, z > -1$ )

$$\frac{1 + x^2}{1 + y + z^2} + \frac{1 + y^2}{1 + z + x^2} + \frac{1 + z^2}{1 + x + y^2} \geq 2$$

**Practice Problem 37.** (USA 2003) ( $a, b, c > 0$ )

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} + \frac{(2b + c + a)^2}{2b^2 + (c + a)^2} + \frac{(2c + a + b)^2}{2c^2 + (a + b)^2} \leq 8$$

**Practice Problem 38.** (Russia 2002) ( $x + y + z = 3$ ,  $x, y, z > 0$ )

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx$$

**Practice Problem 39.** (Latvia 2002) ( $\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1$ ,  $a, b, c, d > 0$ )

$$abcd \geq 3$$

**Practice Problem 40.** (Albania 2002) ( $a, b, c > 0$ )

$$\frac{1 + \sqrt{3}}{3\sqrt{3}}(a^2 + b^2 + c^2) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq a + b + c + \sqrt{a^2 + b^2 + c^2}$$

**Practice Problem 41.** (Belarus 2002) ( $a, b, c, d > 0$ )

$$\sqrt{(a+c)^2 + (b+d)^2} + \frac{2|ad - bc|}{\sqrt{(a+c)^2 + (b+d)^2}} \geq \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \geq \sqrt{(a+c)^2 + (b+d)^2}$$



**Practice Problem 42.** (Canada 2002) ( $a, b, c > 0$ )

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

**Practice Problem 43.** (Vietnam 2002, Dung Tran Nam) ( $a^2 + b^2 + c^2 = 9$ ,  $a, b, c \in \mathbb{R}$ )

$$2(a + b + c) - abc \leq 10$$

**Practice Problem 44.** (Bosnia and Hercegovina 2002) ( $a^2 + b^2 + c^2 = 1$ ,  $a, b, c \in \mathbb{R}$ )

$$\frac{a^2}{1 + 2bc} + \frac{b^2}{1 + 2ca} + \frac{c^2}{1 + 2ab} \leq \frac{3}{5}$$

**Practice Problem 45.** (Junior BMO 2002) ( $a, b, c > 0$ )

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}$$

**Practice Problem 46.** (Greece 2002) ( $a^2 + b^2 + c^2 = 1$ ,  $a, b, c > 0$ )

$$\frac{a}{b^2 + 1} + \frac{b}{c^2 + 1} + \frac{c}{a^2 + 1} \geq \frac{3}{4} \left( a\sqrt{a} + b\sqrt{b} + c\sqrt{c} \right)^2$$

**Practice Problem 47.** (Greece 2002) ( $bc \neq 0$ ,  $\frac{1-c^2}{bc} \geq 0$ ,  $a, b, c \in \mathbb{R}$ )

$$10(a^2 + b^2 + c^2 - bc^3) \geq 2ab + 5ac$$

**Practice Problem 48.** (Taiwan 2002) ( $a, b, c, d \in (0, \frac{1}{2}]$ )

$$\frac{abcd}{(1-a)(1-b)(1-c)(1-d)} \leq \frac{a^4 + b^4 + c^4 + d^4}{(1-a)^4 + (1-b)^4 + (1-c)^4 + (1-d)^4}$$

**Practice Problem 49.** (APMO 2002) ( $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ ,  $x, y, z > 0$ )

$$\sqrt{x+yz} + \sqrt{y+zx} + \sqrt{z+xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}$$

**Practice Problem 50.** (Ireland 2001) ( $x + y = 2$ ,  $x, y \geq 0$ )

$$x^2 y^2 (x^2 + y^2) \leq 2.$$

**Practice Problem 51.** (BMO 2001) ( $a + b + c \geq abc$ ,  $a, b, c \geq 0$ )

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc$$

**Practice Problem 52.** (USA 2001) ( $a^2 + b^2 + c^2 + abc = 4$ ,  $a, b, c \geq 0$ )

$$0 \leq ab + bc + ca - abc \leq 2$$

**Practice Problem 53.** (Columbia 2001) ( $x, y \in \mathbb{R}$ )

$$3(x + y + 1)^2 + 1 \geq 3xy$$

**Practice Problem 54.** (KMO Winter Program Test 2001) ( $a, b, c > 0$ )

$$\sqrt{(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)} \geq abc + \sqrt[3]{(a^3 + abc)(b^3 + abc)(c^3 + abc)}$$

**Practice Problem 55.** (IMO 2001) ( $a, b, c > 0$ )

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

## 4.2. Years 1996 ~ 2000.

**Practice Problem 56.** (IMO 2000, Titu Andreescu) ( $abc = 1$ ,  $a, b, c > 0$ )

$$\left(a - 1 + \frac{1}{b}\right) \left(b - 1 + \frac{1}{c}\right) \left(c - 1 + \frac{1}{a}\right) \leq 1$$

**Practice Problem 57.** (Czech and Slovakia 2000) ( $a, b > 0$ )

$$\sqrt[3]{2(a+b) \left(\frac{1}{a} + \frac{1}{b}\right)} \geq \sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}$$

**Practice Problem 58.** (Hong Kong 2000) ( $abc = 1$ ,  $a, b, c > 0$ )

$$\frac{1+ab^2}{c^3} + \frac{1+bc^2}{a^3} + \frac{1+ca^2}{b^3} \geq \frac{18}{a^3+b^3+c^3}$$

**Practice Problem 59.** (Czech Republic 2000) ( $m, n \in \mathbb{N}$ ,  $x \in [0, 1]$ )

$$(1-x^n)^m + (1-(1-x)^m)^n \geq 1$$

**Practice Problem 60.** (Macedonia 2000) ( $x, y, z > 0$ )

$$x^2 + y^2 + z^2 \geq \sqrt{2}(xy + yz)$$

**Practice Problem 61.** (Russia 1999) ( $a, b, c > 0$ )

$$\frac{a^2+2bc}{b^2+c^2} + \frac{b^2+2ca}{c^2+a^2} + \frac{c^2+2ab}{a^2+b^2} > 3$$

**Practice Problem 62.** (Belarus 1999) ( $a^2 + b^2 + c^2 = 3$ ,  $a, b, c > 0$ )

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{3}{2}$$

**Practice Problem 63.** (Czech-Slovak Match 1999) ( $a, b, c > 0$ )

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1$$

**Practice Problem 64.** (Moldova 1999) ( $a, b, c > 0$ )

$$\frac{ab}{c(c+a)} + \frac{bc}{a(a+b)} + \frac{ca}{b(b+c)} \geq \frac{a}{c+a} + \frac{b}{b+a} + \frac{c}{c+b}$$

**Practice Problem 65.** (United Kingdom 1999) ( $p+q+r = 1, p, q, r > 0$ )

$$7(pq + qr + rp) \leq 2 + 9pqr$$

**Practice Problem 66.** (Canada 1999) ( $x + y + z = 1, x, y, z \geq 0$ )

$$x^2y + y^2z + z^2x \leq \frac{4}{27}$$

**Practice Problem 67.** (Proposed for 1999 USAMO, [AB, pp.25])  
( $x, y, z > 1$ )

$$x^{x^2+2yz} y^{y^2+2zx} z^{z^2+2xy} \geq (xyz)^{xy+yz+zx}$$

**Practice Problem 68.** (Turkey, 1999) ( $c \geq b \geq a \geq 0$ )

$$(a + 3b)(b + 4c)(c + 2a) \geq 60abc$$

**Practice Problem 69.** (Macedonia 1999) ( $a^2 + b^2 + c^2 = 1, a, b, c > 0$ )

$$a + b + c + \frac{1}{abc} \geq 4\sqrt{3}$$

**Practice Problem 70.** (Poland 1999) ( $a + b + c = 1, a, b, c > 0$ )

$$a^2 + b^2 + c^2 + 2\sqrt{3abc} \leq 1$$

**Practice Problem 71.** (Canda 1999) ( $x + y + z = 1, x, y, z \geq 0$ )

$$x^2y + y^2z + z^2x \leq \frac{4}{27}$$

**Practice Problem 72.** (Iran 1998) ( $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2, x, y, z > 1$ )

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

**Practice Problem 73.** (Belarus 1998, I. Gorodnin) ( $a, b, c > 0$ )

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{b+c} + \frac{b+c}{a+b} + 1$$

**Practice Problem 74.** (APMO 1998) ( $a, b, c > 0$ )

$$\left(1 + \frac{a}{b}\right) \left(1 + \frac{b}{c}\right) \left(1 + \frac{c}{a}\right) \geq 2 \left(1 + \frac{a+b+c}{\sqrt[3]{abc}}\right)$$

**Practice Problem 75.** (Poland 1998) ( $a + b + c + d + e + f = 1$ ,  $ace + bdf \geq \frac{1}{108}$ ,  $a, b, c, d, e, f$ )

$$abc + bcd + cde + def + efa + fab \leq \frac{1}{36}$$

**Practice Problem 76.** (Korea 1998) ( $x + y + z = xyz$ ,  $x, y, z > 0$ )

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}$$

**Practice Problem 77.** (Hong Kong 1998) ( $a, b, c \geq 1$ )

$$\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} \leq \sqrt{c(ab+1)}$$

**Practice Problem 78.** (IMO Short List 1998) ( $xyz = 1$ ,  $x, y, z > 0$ )

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

**Practice Problem 79.** (Belarus 1997) ( $a, x, y, z > 0$ )

$$\frac{a+y}{a+x}x + \frac{a+z}{a+x}y + \frac{a+x}{a+y}z \geq x + y + z \geq \frac{a+z}{a+z}x + \frac{a+x}{a+y}y + \frac{a+y}{a+z}z$$

**Practice Problem 80.** (Ireland 1997) ( $a + b + c \geq abc$ ,  $a, b, c \geq 0$ )

$$a^2 + b^2 + c^2 \geq abc$$

**Practice Problem 81.** (Iran 1997) ( $x_1x_2x_3x_4 = 1$ ,  $x_1, x_2, x_3, x_4 > 0$ )

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq \max \left( x_1 + x_2 + x_3 + x_4, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \right)$$

**Practice Problem 82.** (Hong Kong 1997) ( $x, y, z > 0$ )

$$\frac{3 + \sqrt{3}}{9} \geq \frac{xyz(x + y + z + \sqrt{x^2 + y^2 + z^2})}{(x^2 + y^2 + z^2)(xy + yz + zx)}$$

**Practice Problem 83.** (Belarus 1997) ( $a, b, c > 0$ )

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{c+a} + \frac{b+c}{a+b} + \frac{c+a}{b+c}$$

**Practice Problem 84.** (Bulgaria 1997) ( $abc = 1, a, b, c > 0$ )

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}$$

**Practice Problem 85.** (Romania 1997) ( $xyz = 1, x, y, z > 0$ )

$$\frac{x^9 + y^9}{x^6 + x^3y^3 + y^6} + \frac{y^9 + z^9}{y^6 + y^3z^3 + z^6} + \frac{z^9 + x^9}{z^6 + z^3x^3 + x^6} \geq 2$$

**Practice Problem 86.** (Romania 1997) ( $a, b, c > 0$ )

$$\frac{a^2}{a^2 + 2bc} + \frac{b^2}{b^2 + 2ca} + \frac{c^2}{c^2 + 2ab} \geq 1 \geq \frac{bc}{a^2 + 2bc} + \frac{ca}{b^2 + 2ca} + \frac{ab}{c^2 + 2ab}$$

**Practice Problem 87.** (USA 1997) ( $a, b, c > 0$ )

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}$$

**Practice Problem 88.** (Japan 1997) ( $a, b, c > 0$ )

$$\frac{(b+c-a)^2}{(b+c)^2 + a^2} + \frac{(c+a-b)^2}{(c+a)^2 + b^2} + \frac{(a+b-c)^2}{(a+b)^2 + c^2} \geq \frac{3}{5}$$

**Practice Problem 89.** (Estonia 1997) ( $x, y \in \mathbb{R}$ )

$$x^2 + y^2 + 1 > x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}$$

**Practice Problem 90.** (APMC 1996) ( $x + y + z + t = 0, x^2 + y^2 + z^2 + t^2 = 1, x, y, z, t \in \mathbb{R}$ )

$$-1 \leq xy + yz + zt + tx \leq 0$$

**Practice Problem 91.** (Spain 1996) ( $a, b, c > 0$ )

$$a^2 + b^2 + c^2 - ab - bc - ca \geq 3(a - b)(b - c)$$

**Practice Problem 92.** (IMO Short List 1996) ( $abc = 1, a, b, c > 0$ )

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1$$

**Practice Problem 93.** (Poland 1996) ( $a + b + c = 1, a, b, c \geq -\frac{3}{4}$ )

$$\frac{a}{a^2 + 1} + \frac{b}{b^2 + 1} + \frac{c}{c^2 + 1} \leq \frac{9}{10}$$

**Practice Problem 94.** (Hungary 1996) ( $a + b = 1, a, b > 0$ )

$$\frac{a^2}{a + 1} + \frac{b^2}{b + 1} \geq \frac{1}{3}$$

**Practice Problem 95.** (Vietnam 1996) ( $a, b, c \in \mathbb{R}$ )

$$(a + b)^4 + (b + c)^4 + (c + a)^4 \geq \frac{4}{7}(a^4 + b^4 + c^4)$$

**Practice Problem 96.** (Bearus 1996) ( $x + y + z = \sqrt{xyz}, x, y, z > 0$ )

$$xy + yz + zx \geq 9(x + y + z)$$

**Practice Problem 97.** (Iran 1996) ( $a, b, c > 0$ )

$$(ab + bc + ca) \left( \frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} \right) \geq \frac{9}{4}$$

**Practice Problem 98.** (Vietnam 1996) ( $2(ab + ac + ad + bc + bd + cd) + abc + bcd + cda + dab = 16, a, b, c, d \geq 0$ )

$$a + b + c + d \geq \frac{2}{3}(ab + ac + ad + bc + bd + cd)$$

## 4.3. Years 1990 ~ 1995.

**Practice Problem 99.** (Baltic Way 1995) ( $a, b, c, d > 0$ )

$$\frac{a+c}{a+b} + \frac{b+d}{b+c} + \frac{c+a}{c+d} + \frac{d+b}{d+a} \geq 4$$

**Practice Problem 100.** (Canda 1995) ( $a, b, c > 0$ )

$$a^a b^b c^c \geq abc^{\frac{a+b+c}{3}}$$

**Practice Problem 101.** (IMO 1995, Nazar Agakhanov) ( $abc = 1$ ,  $a, b, c > 0$ )

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

**Practice Problem 102.** (Russia 1995) ( $x, y > 0$ )

$$\frac{1}{xy} \geq \frac{x}{x^4+y^2} + \frac{y}{y^4+x^2}$$

**Practice Problem 103.** (Macedonia 1995) ( $a, b, c > 0$ )

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \geq 2$$

**Practice Problem 104.** (APMC 1995) ( $m, n \in \mathbb{N}$ ,  $x, y > 0$ )

$$(n-1)(m-1)(x^{n+m}+y^{n+m})+(n+m-1)(x^n y^m+x^m y^n) \geq nm(x^{n+m-1}y+xy^{n+m-1})$$

**Practice Problem 105.** (Hong Kong 1994) ( $xy+yz+zx = 1$ ,  $x, y, z > 0$ )

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) \leq \frac{4\sqrt{3}}{9}$$

**Practice Problem 106.** (IMO Short List 1993) ( $a, b, c, d > 0$ )

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}$$



**Practice Problem 107.** (APMC 1993) ( $a, b \geq 0$ )

$$\left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2 \leq \frac{a + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + b}{4} \leq \frac{a + \sqrt{ab} + b}{3} \leq \sqrt{\left(\frac{\sqrt[3]{a^2} + \sqrt[3]{b^2}}{2}\right)^3}$$

**Practice Problem 108.** (Poland 1993) ( $x, y, u, v > 0$ )

$$\frac{xy + xv + uy + uv}{x + y + u + v} \geq \frac{xy}{x + y} + \frac{uv}{u + v}$$

**Practice Problem 109.** (IMO Short List 1993) ( $a + b + c + d = 1$ ,  $a, b, c, d > 0$ )

$$abc + bcd + cda + dab \leq \frac{1}{27} + \frac{176}{27}abcd$$

**Practice Problem 110.** (Italy 1993) ( $0 \leq a, b, c \leq 1$ )

$$a^2 + b^2 + c^2 \leq a^2b + b^2c + c^2a + 1$$

**Practice Problem 111.** (Poland 1992) ( $a, b, c \in \mathbb{R}$ )

$$(a+b-c)^2(b+c-a)^2(c+a-b)^2 \geq (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

**Practice Problem 112.** (Vietnam 1991) ( $x \geq y \geq z > 0$ )

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2$$

**Practice Problem 113.** (Poland 1991) ( $x^2 + y^2 + z^2 = 2$ ,  $x, y, z \in \mathbb{R}$ )

$$x + y + z \leq 2 + xyz$$

**Practice Problem 114.** (Mongolia 1991) ( $a^2 + b^2 + c^2 = 2$ ,  $a, b, c \in \mathbb{R}$ )

$$|a^3 + b^3 + c^3 - abc| \leq 2\sqrt{2}$$

**Practice Problem 115.** (IMO Short List 1990) ( $ab + bc + cd + da = 1$ ,  $a, b, c, d > 0$ )

$$\frac{a^3}{b + c + d} + \frac{b^3}{c + d + a} + \frac{c^3}{d + a + b} + \frac{d^3}{a + b + c} \geq \frac{1}{3}$$

## 4.4. Supplementary Problems.

**Practice Problem 116.** (Lithuania 1987) ( $x, y, z > 0$ )

$$\frac{x^3}{x^2 + xy + y^2} + \frac{y^3}{y^2 + yz + z^2} + \frac{z^3}{z^2 + zx + x^2} \geq \frac{x + y + z}{3}$$

**Practice Problem 117.** (Yugoslavia 1987) ( $a, b > 0$ )

$$\frac{1}{2}(a + b)^2 + \frac{1}{4}(a + b) \geq a\sqrt{b} + b\sqrt{a}$$

**Practice Problem 118.** (Yugoslavia 1984) ( $a, b, c, d > 0$ )

$$\frac{a}{b + c} + \frac{b}{c + d} + \frac{c}{d + a} + \frac{d}{a + b} \geq 2$$

**Practice Problem 119.** (IMO 1984) ( $x + y + z = 1, x, y, z \geq 0$ )

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$$

**Practice Problem 120.** (USA 1980) ( $a, b, c \in [0, 1]$ )

$$\frac{a}{b + c + 1} + \frac{b}{c + a + 1} + \frac{c}{a + b + 1} + (1 - a)(1 - b)(1 - c) \leq 1.$$

**Practice Problem 121.** (USA 1979) ( $x + y + z = 1, x, y, z > 0$ )

$$x^3 + y^3 + z^3 + 6xyz \geq \frac{1}{4}.$$

**Practice Problem 122.** (IMO 1974) ( $a, b, c, d > 0$ )

$$1 < \frac{a}{a + b + d} + \frac{b}{b + c + a} + \frac{c}{b + c + d} + \frac{d}{a + c + d} < 2$$

**Practice Problem 123.** (IMO 1968) ( $x_1, x_2 > 0, y_1, y_2, z_1, z_2 \in \mathbb{R}, x_1y_1 > z_1^2, x_2y_2 > z_2^2$ )

$$\frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2} \geq \frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2}$$

**Practice Problem 124.** (Nesbitt's inequality) ( $a, b, c > 0$ )

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

**Practice Problem 125.** (Polya's inequality) ( $a \neq b, a, b > 0$ )

$$\frac{1}{3} \left( 2\sqrt{ab} + \frac{a+b}{2} \right) \geq \frac{\ln b - \ln a}{b-a}$$

**Practice Problem 126.** (Klamkin's inequality) ( $-1 < x, y, z < 1$ )

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \geq 2$$

**Practice Problem 127.** (Carlson's inequality) ( $a, b, c > 0$ )

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}} \geq \sqrt{\frac{ab+bc+ca}{3}}$$

**Practice Problem 128.** ([ONI], Vasile Cirtoaje) ( $a, b, c > 0$ )

$$\left(a + \frac{1}{b} - 1\right) \left(b + \frac{1}{c} - 1\right) + \left(b + \frac{1}{c} - 1\right) \left(c + \frac{1}{a} - 1\right) + \left(c + \frac{1}{a} - 1\right) \left(a + \frac{1}{b} - 1\right) \geq 3$$

**Practice Problem 129.** ([ONI], Vasile Cirtoaje) ( $a, b, c, d > 0$ )

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geq 0$$

**Practice Problem 130.** (Elemente der Mathematik, Problem 1207, Šefket Arslanagić) ( $x, y, z > 0$ )

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{x+y+z}{\sqrt[3]{xyz}}$$

**Practice Problem 131.** ( $\sqrt{WURZEL}$ , Walther Janous) ( $x+y+z = 1, x, y, z > 0$ )

$$(1+x)(1+y)(1+z) \geq (1-x^2)^2 + (1-y^2)^2 + (1-z^2)^2$$

**Practice Problem 132.** ( $\sqrt{WURZEL}$ , Heinz-Jürgen Seiffert) ( $xy > 0, x, y \in \mathbb{R}$ )

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy} + \frac{x+y}{2}$$

**Practice Problem 133.** ( $\sqrt{WURZEL}$ , Šefket Arslanagić) ( $a, b, c > 0$ )

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}$$

**Practice Problem 134.** ( $\sqrt{WURZEL}$ , Šefket Arslanagić) ( $abc = 1, a, b, c > 0$ )

$$\frac{1}{a^2(b+c)} + \frac{1}{b^2(c+a)} + \frac{1}{c^2(a+b)} \geq \frac{3}{2}.$$

**Practice Problem 135.** ( $\sqrt{WURZEL}$ , Peter Starek, Donauwörth) ( $abc = 1, a, b, c > 0$ )

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq \frac{1}{2}(a+b)(c+a)(b+c) - 1.$$

**Practice Problem 136.** ( $\sqrt{WURZEL}$ , Peter Starek, Donauwörth) ( $x+y+z=3, x^2+y^2+z^2=7, x, y, z > 0$ )

$$1 + \frac{6}{xyz} \geq \frac{1}{3} \left( \frac{x}{z} + \frac{y}{x} + \frac{z}{y} \right)$$

**Practice Problem 137.** ( $\sqrt{WURZEL}$ , Šefket Arslanagić) ( $a, b, c > 0$ )

$$\frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1} \geq \frac{3(a+b+c)}{a+b+c+3}.$$

**Practice Problem 138.** ([ONI], Gabriel Dospinescu, Mircea Lascu, Marian Tetiva) ( $a, b, c > 0$ )

$$a^2 + b^2 + c^2 + 2abc + 3 \geq (1+a)(1+b)(1+c)$$

**Practice Problem 139.** (Gazeta Matematică) ( $a, b, c > 0$ )

$$\sqrt{a^4 + a^2b^2 + b^4} + \sqrt{b^4 + b^2c^2 + c^4} + \sqrt{c^4 + c^2a^2 + a^4} \geq a\sqrt{2a^2 + bc} + b\sqrt{2b^2 + ca} + c\sqrt{2c^2 + ab}$$

**Practice Problem 140.** (C<sup>2</sup>2362, Mohammed Aassila) ( $a, b, c > 0$ )

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{3}{1+abc}$$

**Practice Problem 141.** (C2580) ( $a, b, c > 0$ )

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{b+c}{a^2+bc} + \frac{c+a}{b^2+ca} + \frac{a+b}{c^2+ab}$$

**Practice Problem 142.** (C2581) ( $a, b, c > 0$ )

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq a+b+c$$

**Practice Problem 143.** (C2532) ( $a^2 + b^2 + c^2 = 1$ ,  $a, b, c > 0$ )

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2(a^3 + b^3 + c^3)}{abc}$$

**Practice Problem 144.** (C3032, Vasile Cirtoaje) ( $a^2 + b^2 + c^2 = 1$ ,  $a, b, c > 0$ )

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ca} \leq \frac{9}{2}$$

**Practice Problem 145.** (C2645) ( $a, b, c > 0$ )

$$\frac{2(a^3 + b^3 + c^3)}{abc} + \frac{9(a+b+c)^2}{(a^2 + b^2 + c^2)} \geq 33$$

**Practice Problem 146.** ( $x, y \in \mathbb{R}$ )

$$-\frac{1}{2} \leq \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \leq \frac{1}{2}$$

**Practice Problem 147.** ( $0 < x, y < 1$ )

$$x^y + y^x > 1$$

**Practice Problem 148.** ( $x, y, z > 0$ )

$$\sqrt[3]{xyz} + \frac{|x-y| + |y-z| + |z-x|}{3} \geq \frac{x+y+z}{3}$$

**Practice Problem 149.** ( $a, b, c, x, y, z > 0$ )

$$\sqrt[3]{(a+x)(b+y)(c+z)} \geq \sqrt[3]{abc} + \sqrt[3]{xyz}$$

**Practice Problem 150.** ( $x, y, z > 0$ )

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(y+z)(y+x)}} + \frac{z}{z + \sqrt{(z+x)(z+y)}} \leq 1$$

**Practice Problem 151.** ( $x + y + z = 1, x, y, z > 0$ )

$$\frac{x}{\sqrt{1-x}} + \frac{y}{\sqrt{1-y}} + \frac{z}{\sqrt{1-z}} \geq \sqrt{\frac{3}{2}}$$

**Practice Problem 152.** ( $a, b, c \in \mathbb{R}$ )

$$\sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \geq \frac{3\sqrt{2}}{2}$$

**Practice Problem 153.** ( $a, b, c > 0$ )

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

**Practice Problem 154.** ( $xy + yz + zx = 1, x, y, z > 0$ )

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \geq \frac{2x(1-x^2)}{(1+x^2)^2} + \frac{2y(1-y^2)}{(1+y^2)^2} + \frac{2z(1-z^2)}{(1+z^2)^2}$$

**Practice Problem 155.** ( $x, y, z \geq 0$ )

$$xyz \geq (y+z-x)(z+x-y)(x+y-z)$$

**Practice Problem 156.** ( $a, b, c > 0$ )

$$\sqrt{ab(a+b)} + \sqrt{bc(b+c)} + \sqrt{ca(c+a)} \geq \sqrt{4abc + (a+b)(b+c)(c+a)}$$

**Practice Problem 157.** (Darij Grinberg) ( $x, y, z \geq 0$ )

$$\left( \sqrt{x(y+z)} + \sqrt{y(z+x)} + \sqrt{z(x+y)} \right) \cdot \sqrt{x+y+z} \geq 2\sqrt{(y+z)(z+x)(x+y)}.$$

**Practice Problem 158.** (Darij Grinberg) ( $x, y, z > 0$ )

$$\frac{\sqrt{y+z}}{x} + \frac{\sqrt{z+x}}{y} + \frac{\sqrt{x+y}}{z} \geq \frac{4(x+y+z)}{\sqrt{(y+z)(z+x)(x+y)}}.$$

**Practice Problem 159.** (Darij Grinberg) ( $a, b, c > 0$ )

$$\frac{a^2(b+c)}{(b^2+c^2)(2a+b+c)} + \frac{b^2(c+a)}{(c^2+a^2)(2b+c+a)} + \frac{c^2(a+b)}{(a^2+b^2)(2c+a+b)} > \frac{2}{3}.$$

**Practice Problem 160.** (Darij Grinberg) ( $a, b, c > 0$ )

$$\frac{a^2}{2a^2 + (b+c)^2} + \frac{b^2}{2b^2 + (c+a)^2} + \frac{c^2}{2c^2 + (a+b)^2} < \frac{2}{3}.$$

**Practice Problem 161.** (Vasile Cirtoaje) ( $a, b, c \in \mathbb{R}$ )

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a)$$

## 5. APPENDIX

## 5.1. References.

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5.2. **IMO Code.**

AFG	Afghanistan	ALB	Albania	ALG	Algeria
ARG	Argentina	ARM	Armenia	AUS	Australia
AUT	Austria	AZE	Azerbaijan	BAH	Bahrain
BGD	Bangladesh	BLR	Belarus	BEL	Belgium
BEN	Benin	BOL	Bolivia	BIH	BIH
BRA	Brazil	BRU	Brunei	BGR	Bulgaria
KHM	Cambodia	CMR	Cameroon	CAN	Canada
CHI	Chile	CHN	CHN	COL	Colombia
CIS	CIS	CRI	Costa Rica	HRV	Croatia
CUB	Cuba	CYP	Cyprus	CZE	Czech Republic
CZS	Czechoslovakia	DEN	Denmark	DOM	Dominican Republic
ECU	Ecuador	EST	Estonia	FIN	Finland
FRA	France	GEO	Georgia	GDR	GDR
GER	Germany	HEL	Greece	GTM	Guatemala
HND	Honduras	HKG	Hong Kong	HUN	Hungary
ISL	Iceland	IND	India	IDN	Indonesia
IRN	Islamic Republic of Iran	IRL	Ireland	ISR	Israel
ITA	Italy	JPN	Japan	KAZ	Kazakhstan
PRK	PRK	KOR	Republic of Korea	KWT	Kuwait
KGZ	Kyrgyzstan	LVA	Latvia	LIE	Liechtenstein
LTU	Lithuania	LUX	Luxembourg	MAC	Macau
MKD	MKD	MAS	Malaysia	MLT	Malta
MRT	Mauritania	MEX	Mexico	MDA	Republic of Moldova
MNG	Mongolia	MNE	Montenegro	MAR	Morocco
MOZ	Mozambique	NLD	Netherlands	NZL	New Zealand
NIC	Nicaragua	NGA	Nigeria	NOR	Norway
PAK	Pakistan	PAN	Panama	PAR	Paraguay
PER	Peru	PHI	Philippines	POL	Poland
POR	Portugal	PRI	Puerto Rico	ROU	Romania
RUS	Russian Federation	SLV	El Salvador	SAU	Saudi Arabia
SEN	Senegal	SRB	Serbia	SCG	Serbia and Montenegro
SGP	Singapore	SVK	Slovakia	SVN	Slovenia
SAF	South Africa	ESP	Spain	LKA	Sri Lanka
SWE	Sweden	SUI	Switzerland	SYR	Syria
TWN	Taiwan	TJK	Tajikistan	THA	Thailand
TTO	Trinidad and Tobago	TUN	Tunisia	TUR	Turkey
NCY	NCY	TKM	Turkmenistan	UKR	Ukraine
UAE	United Arab Emirates	UNK	United Kingdom	USA	United States of America
URY	Uruguay	USS	USS	UZB	Uzbekistan
VEN	Venezuela	VNM	Vietnam	YUG	Yugoslavia

## THE PURSUIT OF JOY-I

BIH	Bosnia and Herzegovina
CHN	People's Republic of China
CIS	Commonwealth of Independent States
FRG	Federal Republic of Germany
GDR	German Democratic Republic
MKD	The Former Yugoslav Republic of Macedonia
NCY	Turkish Republic of Northern Cyprus
PRK	Democratic People's Republic of Korea
USS	Union of the Soviet Socialist Republics