A Note on the Irrationality of $\zeta(3)$

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Abstract. We give an account of F. Beukers' [2] proof of the irrationality of $\zeta(3)$.

Key Words: Zeta functions, Apéry's constant.

1. Introduction

At the "Journees Arithmetiques" held at Marseille–Luminy in June 1978, Roger Apéry gave an elmentary proof of the irrationality of $\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \ldots$.However due to the complexity of the proof there was a general disagreement amongst the mathematicians present there as to the validity of the proof presented. Two months later a complete exposition of the proof was presented at the International Congress of Mathematicians in Helsinki in August 1978 by H. Cohen. This proof was based on the lecture by Apéry but also contained some ideas of Cohen and Don Zaiger. Then in 1979 F. Beukers gave a very simple proof of the result in [2]. In this note we shall give an account of the proof given by Beukers. Beukers' proof uses some elementary calculus involving few double and triple integrals.

2. Preliminaries

We prove in this section few non-trivial lemmas which shall be required in our proof of the irrationality of $\zeta(3)$.

Lemma 2.1. *The LCM of* $1, 2, 3 \ldots, n$ *is denoted by* d_n*, then* $d_n = \prod_{p\leq n} p^{\lfloor \frac{\log n}{\log p} \rfloor} < \prod_{p\leq n} p^{\frac{\log n}{\log p}} = n^{\pi(n)}$.

Proof. If $p_1, p_2, \ldots, p_{\pi(n)}$ are all the prime numbers less than or equal to n. If p is a prime bigger than n then p does not divide any of $1, 2, 3, \ldots, n$, so p doesn't divide d_n. Thus $d_n = p_1^{a_1} \cdot p_2^{a_2} \cdots, p_{\pi(n)}^{a_{\pi(n)}}$ for some a_is ($i = 1, 2, 3, \ldots, \pi(n)$). We have that $p^{\lfloor \log_p n \rfloor} \leq n$ and $p^{\lfloor \log_p n \rfloor + 1} > n$ hold for any prime less or equal than d_n. So $p^{\lfloor \log_p n \rfloor} \mid\mid d_n$. Thus $a_i = \log_{p_i} n$. Hence, $d_n = \prod_{p\leq n} p^{\lfloor \log_p n \rfloor} = \prod_{p\leq n} p^{\lfloor \frac{logn}{logp} \rfloor} < \prod_{p\leq n} p^{\frac{logn}{logp}} = \prod_{p\leq n} p^{\log_p n} = \prod_{p\leq n} n = n^{\pi(n)}$. □

Lemma 2.2. $n^\pi(n) < 3^n$.

Proof. Accoring to the prime number theorem we have $\pi(n) < \frac{(\log 3)n}{\log n}$ for sufficiently large n. So, $n^\pi(n) < 3^n$. □

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Lemma 2.3. *Let r and s be non-negative integers. If $r > s$ then,*

- $\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dxdy$ *is a rational number whose denominator is a divisor of* d_r^2.
- $\int_0^1 \int_0^1 \frac{-\log xy}{1-xy} x^r y^s dxdy$ *is a rational number whose denominator is a divisor of* d_r^3.

If $r = s$, then

- $\int_0^1 \int_0^1 \frac{x^r y^r}{1-xy} dxdy = \zeta(2) - \frac{1}{1^2} - \frac{1}{2^2} - \ldots - \frac{1}{r^2}$.
- $\int_0^1 \int_0^1 \frac{-\log xy}{1-xy} x^r y^r dxdy = 2\{\zeta(3) - \frac{1}{1^3} - \frac{1}{2^3} - \ldots - \frac{1}{r^3}\}$.

When $r = 0$ we let the sums $\frac{1}{1^2} + \ldots + \frac{1}{r^2}$ and $\frac{1}{1^3} + \ldots + \frac{1}{r^3}$ vanish.

Proof. Let σ be any non-negative integer. We consider the integral,

$$I_n = \int_0^1 \int_0^1 \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy} dxdy \tag{2.1}$$

It is very easy to see that when the denominator of the I_n is expressed as a Geometric Series we get the following,

$$I_n = \int_0^1 \int_0^1 \sum_{k=0}^{\infty} (xy)^k x^{r+\sigma} y^{s+\sigma} dxdy$$

$$= \sum_{k=0}^{\infty} (\int_0^1 x^{k+r+\sigma} dx)(\int_0^1 y^{k+s+\sigma} dy)$$

$$= \sum_{k=0}^{\infty} (\frac{1}{k+r+\sigma+1} . \frac{1}{k+s=\sigma+1})$$

$$= \sum_{k+0}^{\infty} (\frac{1}{k+s+\sigma+1} - \frac{1}{k+r+\sigma+1}) \tag{2.2}$$

The above is true since $r > s$. So, finally we have,

$$I_n = \frac{1}{r-s} (\frac{1}{s+1+\sigma} + \ldots + \frac{1}{r+\sigma}) \tag{2.3}$$

If we put $\sigma = 0$ now then the first part of our lemma follows immediately. Differentiating with respect to σ and putting $\sigma = 0$, I_n changes to ,

$$I_n = \int_0^1 \int_0^1 \frac{\log xy}{1-xy} x^r y^s dxdy \tag{2.4}$$

And the summation becomes

$$-\frac{1}{r-s} (\frac{1}{(s+1)^2} + \ldots + \frac{1}{r^2}$$

This proves the second part of the lemma.

We now assume $r = s$, then it is easy to see that,

$$I_n = \int_0^1 \int_0^1 \frac{x^{r+\sigma} y^{r+\sigma}}{1 - xy} dxdy = \sum_{k=0}^{\infty} \frac{1}{(k + r + \sigma + 1)^2}$$

By putting $\sigma = 0$ as before we get the third part of the lemma as follows,

$$I_n = \frac{1}{(r+1)^2} + \frac{1}{(r+2)^2} + \dots$$

This equals,

$$(\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{r^2} + \frac{1}{(r+1)^2} + \frac{1}{(r+2)^2} + \dots) - (\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{r^2})$$

Thus,

$$I_n = \zeta(2) - \frac{1}{1^2} - \frac{1}{2^2} - \dots - \frac{1}{r^2}$$

Again, we differentiate with respect to σ and put $\sigma = 0$. Then,

$$I_n = \int_0^1 \int_0^1 \frac{\log xy}{1 - xy} x^r y^r dxdy = \sum_{k=0}^{\infty} \frac{-2}{(k + r + 1)^3}$$

The above equals to

$$-2(\frac{1}{(r+1)^3} + \frac{1}{(r+2)^3} + \dots)$$

which in turn equals to

$$-2((\frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{r^3} + \frac{1}{(r+1)^3} + \dots) - (\frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{r^3}))$$

Thus,

$$I_n = -2(\zeta(3) - \frac{1}{1^3} - \frac{1}{2^3} - \dots - \frac{1}{r^3})$$

which proves the final part of the lemma.

□

Lemma 2.4. $\int_0^1 \frac{1}{1-(1-xy)z} dz = \frac{-\log xy}{1-xy}$.

Proof. We substitute $(1 - xy)z = u$ in the integral which gives $(1 - xy)dz = du$. Changing the limits we get,

$$\int_0^1 \frac{1}{1-(1-xy)z}dz = \int_0^{1-xy} \frac{1}{(1-u)(1-xy)}du$$

$$= \frac{1}{1-xy}\int_o^{1-xy}\frac{1}{1-u}du$$

$$= \frac{1}{1-xy} \mid -\log(1-u) \mid_1^{1-xy}$$

$$= \frac{-\log xy}{1-xy}$$

□

Lemma 2.5. *If* $F(u,v,w) = \frac{u(1-u)v(1-v)w(1-w)}{1-(1-uv)w}$ *and* $u,v,w \in (o,1)$ *then*

$$Max\ F(u,v,w) \leq \frac{1}{27}.$$

Proof. We have

$$1-(1-uv)w = 1-w+uvw \geq 2(\sqrt{(1-w)}\sqrt{uvw})$$

So,

$$F(u,v,w) \leq \frac{1}{2}\sqrt{(1-w)w}\sqrt{u}(1-u)\sqrt{v}(1-v)$$

Again,

$$Max_{0\leq t\leq 1}\ t(1-t^2) = \frac{1}{\sqrt{3}}.\frac{2}{3},\ t = \frac{1}{\sqrt{3}}$$

$$Max_{0\leq t\leq 1}\ \sqrt{t(1-t)} = \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}},\ t = \frac{1}{2}$$

So,

$$Max\ [F(u,v,w)] \leq \frac{1}{2}.\frac{1}{2}.\frac{2}{3\sqrt{3}}.\frac{2}{3\sqrt{3}} = \frac{1}{27}.$$

□

3. Irrationality of $\zeta(3)$

Theorem 3.1. $\zeta(3)$ *is irrational.*

Proof. We consider the integral,

$$I_n = \int_0^1\int_0^1 \frac{-\log xy}{1-xy}P_n(x)P_n(y)dxdy,$$

where $P_n(r) = (\frac{d}{dr})^n r^n(1-r)^n.\frac{1}{n!}$. It is clear from Lemma 2.3 that for some $A_n, B_n \in Z$ we have,

$$I_n = (A_n + B_n\zeta(3))d_n^{-3} \tag{3.1}$$

Using Lemma 2.4 I_n changes into,

$$I_n = \int_0^1 \int_0^1 \int_0^1 \frac{P_n(x)P_n(y)}{1-(1-xy)z} dxdydz$$

After partially integrating the above integral with respect to x n-times the interal changes into,

$$I_n = \int_0^1 \int_0^1 \int_0^1 \frac{(xyz)^n(1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}} dxdydz \tag{3.2}$$

We now use the substitution $w = \frac{1-z}{1-(1-xy)z}$ to get,

$$I_n = \int_0^1 \int_0^1 \int_0^1 (1-x)^n(1-w)^n \frac{P_n(y)}{1-(1-xy)w} dxdydw$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{[x(1-x)y(1-y)w(1-w)]^n}{[1-(1-xy)w]^{n+1}} dxdydw \tag{3.3}$$

where the last equality again follows from an n-fold integration by parts with respect to w like earlier.

From Lemma 2.5 and Lemma 2.4 it follows that the integral is bounded above by

$$(\frac{1}{27})^n \int_0^1 \int_0^1 \int_0^1 \frac{1}{1-(1-xy)w} dxdydw = (\frac{1}{27})^n \int_0^1 \int_0^1 \frac{-\log xy}{1-xy} dxdy$$

The above is equal to $2(\frac{1}{27})^n\zeta(3)$ by Lemma 2.3.

Since integral (3.3) is not zero, so we have

$$0 <\mid A_n + B_n\zeta(3) \mid d_n^{-3} < 2\zeta(3)(\frac{1}{27})^n$$

and hence by Lemma 2.1 and Lemma 2.2

$$0 <\mid A_n + B_n\zeta(3) \mid < 2\zeta(3)d_n^3(\frac{1}{27})^n < 2\zeta(3) < (\frac{4}{5})^n \tag{3.4}$$

for sufficiently large n, which proves the irrationality of $\zeta(3)$.

□

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References

[1] M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, Springer India, 2010.
[2] F. Beukers, *A note on the Irrationality of* $\zeta(2)$ *and* $\zeta(3)$, Bull. Lon. Math. Soc. **11** (1979), 268–272.
[3] F. Beukers, *Elementary Number Theory WISB321*, Lecture Notes, 2009.
[4] J. M. Borwein, P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, Wiley–Intescience, 1998.

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