Tiling Problems and Perfect Matchings

Manjil P. Saikia

Universität Wien

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Figure: A $2 \times n$ board

Tiling Problems

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But not all tiling problems are easy!

Tiling a hexagon

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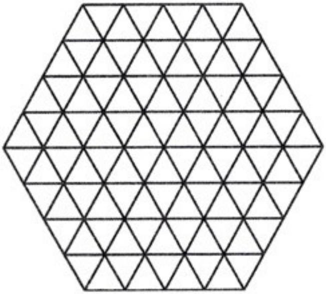

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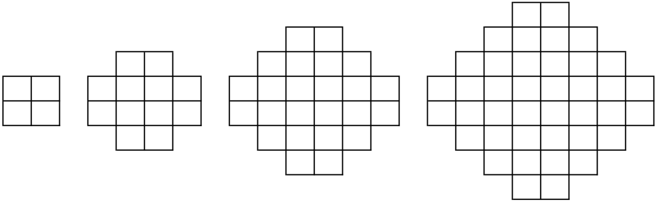

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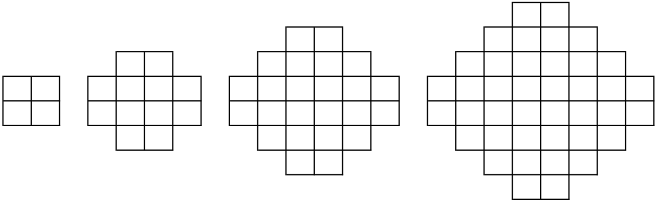

We will focus on these objects, in this talk.

Aztec Diamonds

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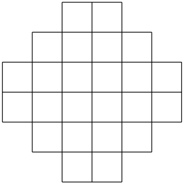

Figure: AD(3), Aztec Diamond of order 3

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Figure: A Mezoamerican pyramid

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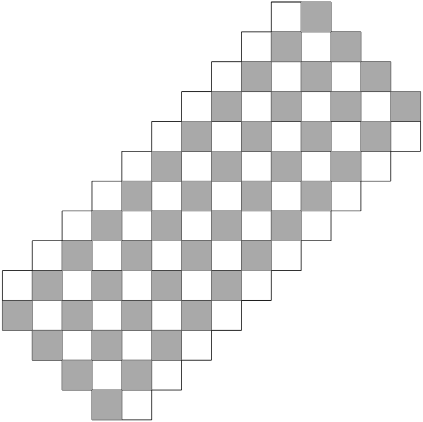

Figure: Checkerboard representation of an Aztec Rectangle

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- For $a < b$, $\mathcal{AR}_{a,b}$ does not have any tiling by dominoes.
- The non-tileability of the region $\mathcal{AR}_{a,b}$ becomes evident if we look at the checkerboard representation of $\mathcal{AR}_{a,b}$

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Theorem (Mills-Robbins-Rumsey)

Let $a < b$ be positive integers and $1 \le s_1 < s_2 < \cdots < s_a \le b$. Then the number of domino tilings of $\mathcal{AR}_{a,b}$ where all unit squares from the south-eastern side are removed except for those in positions $s_1, s_2, \ldots, s_a$ is

$$2^{a(a+1)/2} \prod_{1 \le i < j \le a} \frac{s_j - s_i}{j - i}.$$

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We will show one technique in this talk.

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- $w, x, y, z \in V_1, |V_1| = |V_2| + 2$; first term vanishes

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- There is a blue-red alternating path from w to one of x, y, z
- Two such paths cannot cross, so w does not connect to y
- Switch the edges in the path of w and get a pair of matchings of $G - \{w, x\}$ and $G - \{y, z\}$ or of $G - \{w, z\}$ and $G - \{x, y\}$

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$$\mathrm{Pf}(A) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}.$$

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- Then we define $G + W$ as follows: $G + W$ is the induced subgraph of H with vertex set $\mathrm{V}(G + W) = \mathrm{V}(G) \Delta \mathrm{V}(W)$, where Δ denotes the symmetric difference of sets.

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Then we have that

$$\mathrm{M}(G + \{a_1, a_2, \ldots, a_{2k}\}) = \frac{\mathrm{Pf}(A)}{[\mathrm{M}(G)]^{k-1}}. \tag{1.2}$$

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where $\overline{\{a_i, a_j\}}$ stands for the complement of $\{a_i, a_j\}$ in the set $\{a_1, \dots, a_{2k}\}$.

Back to the beginning

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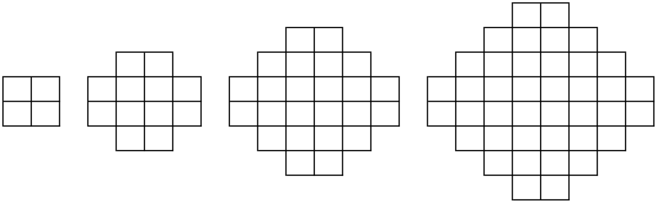

We are interested in counting the number of domino tilings of these regions.

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What is the dual graph?

Dual Graphs

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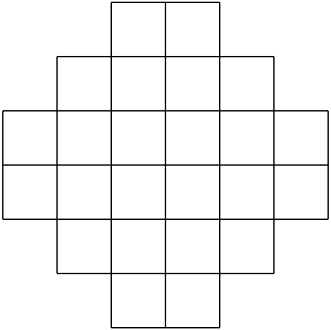

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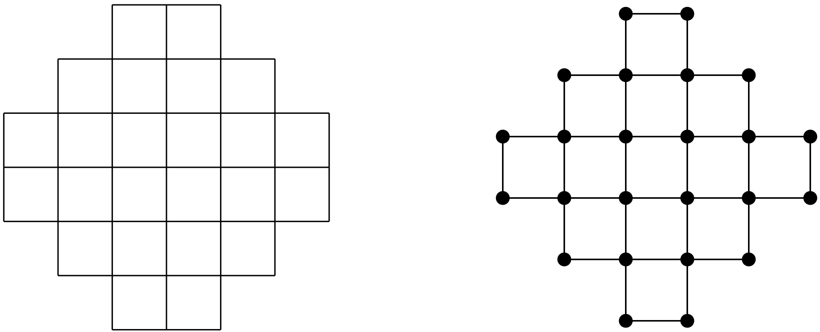

Figure: Aztec Diamond of order 3 and it's dual graph

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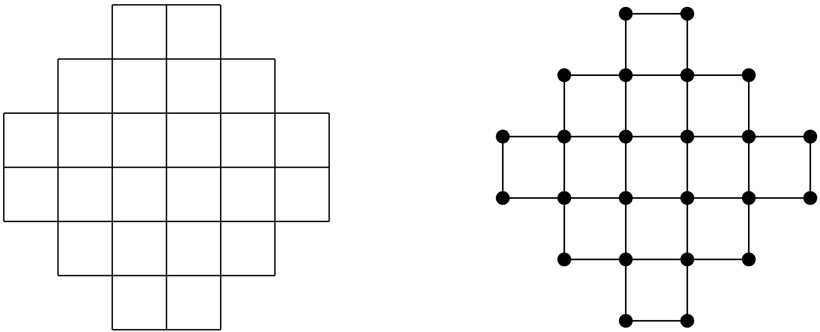

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So, we can use the terms matchings and tilings equivalently.

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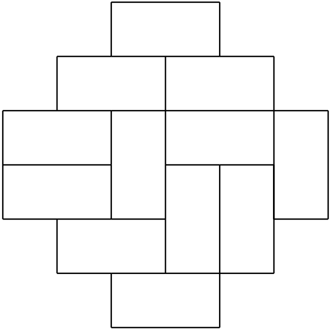

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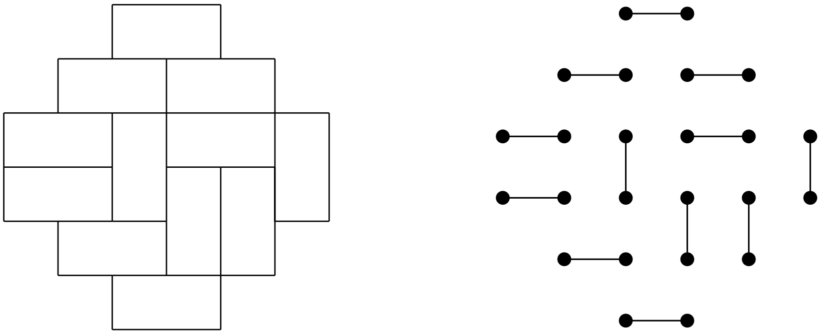

Figure: Equivalence of tilings and matchings

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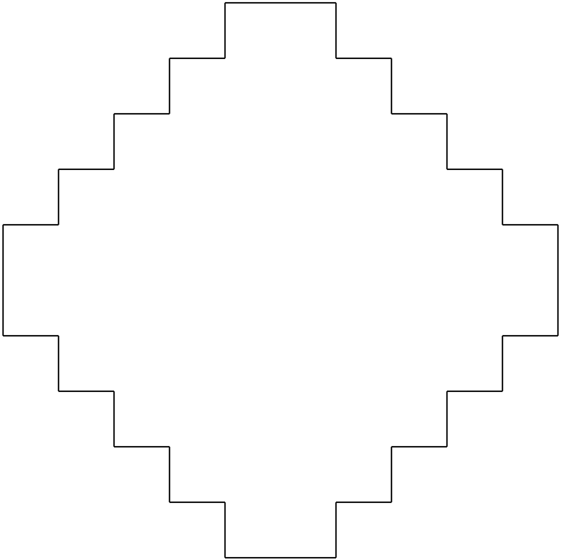

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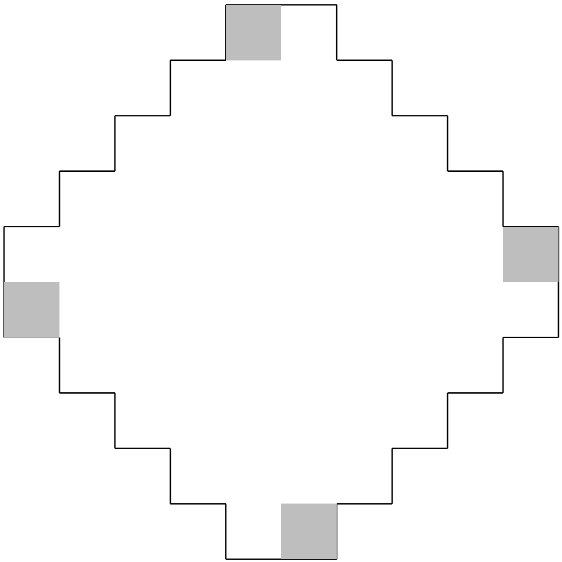

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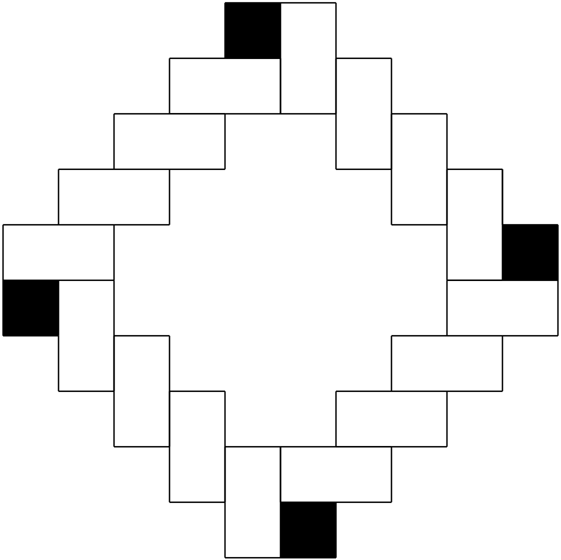

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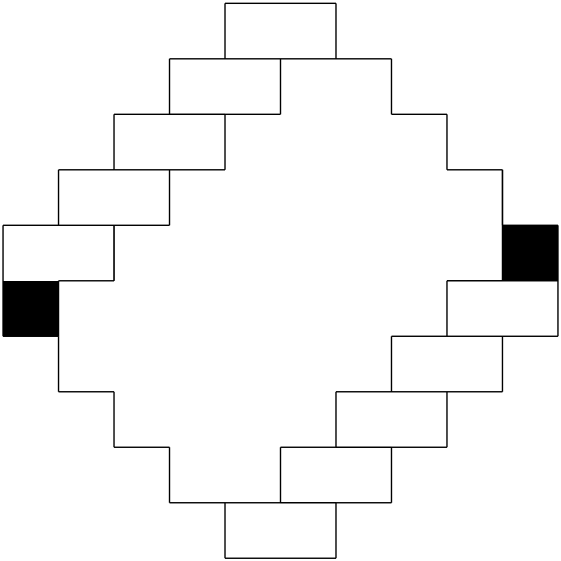

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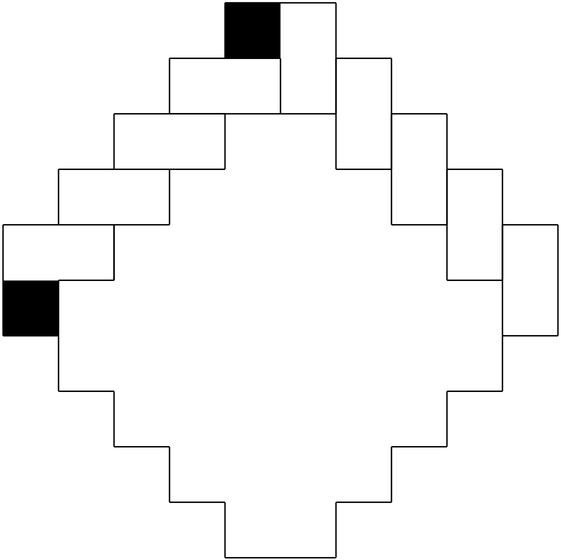

Extensions?

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Extensions?

What about other type of regions?

Extensions?

What about other type of regions?

Or, regions with some holes (defects) on them?

Aztec Diamond with defects on adjacent sides

Aztec Diamond with defects on adjacent sides

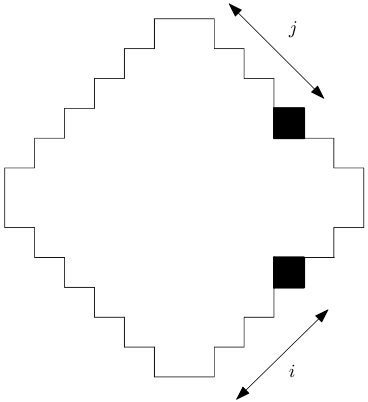

Figure: Aztec Diamond with defects on adjacent sides

Aztec Diamond with defects on adjacent sides

Aztec Diamond with defects on adjacent sides

Proposition

Let a, i, j be positive integers such that $1 \leq i, j \leq a$, then the number of domino tilings of $\mathrm{AD}(a)$ with one defect on the southeastern side at the i-th position counted from the south corner and one defect on the northeastern side on the j-th position counted from the north corner is given by

$$2^{a(a-1)/2} \binom{a-1}{i-1} \binom{a-1}{j-1} {}_3F_2 \left[\begin{matrix} 1, 1-i, 1-j \\ 1-a, 1-a \end{matrix} ; 2 \right].$$

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Here

$${}_rF_s\left[\begin{matrix}a_1, a_2, \ldots, a_r\\ b_1, \ldots, b_s\end{matrix};z\right] = \sum_{k\ge 0}\frac{(a_1)_k(a_2)_k\cdots(a_r)_k}{(b_1)_k(b_2)_k\cdots(b_s)_k}\frac{z^k}{k!}$$

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and

$$(p)_n = p(p+1)(p+2) \cdots (p+n-1).$$

Proof

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We use Kuo condensation, with the vertices marked as follows.

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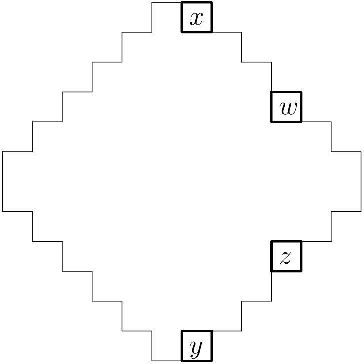

Figure: Aztec Diamond with some marked squares; here $a = 6$

Forced dominoes for different choices of labels

Forced dominoes for different choices of labels

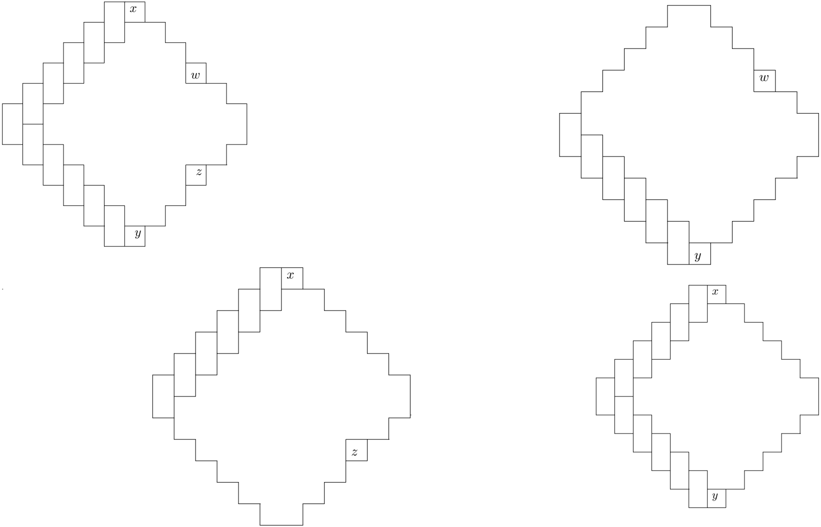

Figure: Forced dominoes, where the vertices we remove are marked

Proof contd.

Proof contd.

$$\begin{aligned} \mathrm{M}(\mathrm{AD}_a(i,j))\,\mathrm{M}(\mathrm{AD}(a-1)) = {} & \mathrm{M}(\mathrm{AD}(a))\,\mathrm{M}(\mathrm{AD}_{a-1}(i-1,j-1)) \\ & + \mathrm{M}(\mathcal{AR}_{a-1,a}(j))\,\mathrm{M}(\mathcal{AR}_{a-1,a}(i)). \end{aligned} \tag{1.3}$$

Proof contd.

$$\mathrm{M}(\mathrm{AD}_a(i,j))\,\mathrm{M}(\mathrm{AD}(a-1)) = \mathrm{M}(\mathrm{AD}(a))\,\mathrm{M}(\mathrm{AD}_{a-1}(i-1,j-1)) + \mathrm{M}(\mathcal{AR}_{a-1,a}(j))\,\mathrm{M}(\mathcal{AR}_{a-1,a}(i)). \tag{1.3}$$

$$\mathrm{M}(\mathrm{AD}_a(i,j)) = 2^a\,\mathrm{M}(\mathrm{AD}_{a-1}(i-1,j-1)) + 2^{a(a-1)/2}\binom{a-1}{j-1}\binom{a-1}{i-1} \tag{1.4}$$

Proof contd.

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Now, we use induction to get the result.

More holes?

More holes?

But, what about arbitrary holes?

More holes?

But, what about arbitrary holes?

On the boundary?

Regions with defects

Regions with defects

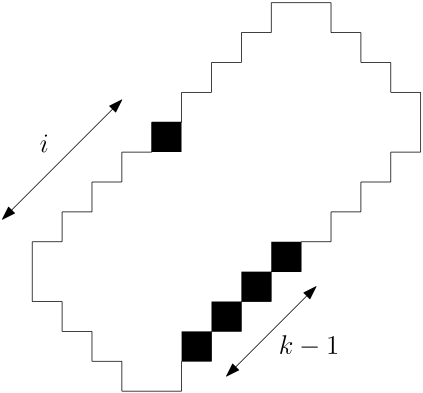

Figure: An $a \times b$ Aztec rectangle with defects marked in black; here $a = 4, b = 9.k = 5, i = 5$

Regions with defects

Regions with defects

Proposition

Let $1 \leq a, i \leq b$ be positive integers with $k = b - a > 0$, then the number of domino tilings of $\mathcal{AR}_{a,b}(2, 3, \ldots, k)$ with a defect on the northwestern side in the i-th position counted from the west corner is given by

$$2^{a(a+1)/2} \binom{a+k-2}{k-1} \binom{a}{a-i+k} {}_3F_2 \left[\begin{matrix} 1, -k-1, i-a-k \\ i-k+1, 2-a-k \end{matrix} ; -1 \right].$$

Preliminaries

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We define the region $\mathcal{AR}^k_{a,b}$ to be the region obtained from $\mathcal{AR}_{a.b}$ by adding a string of k unit squares along the boundary of the southeastern side (γ defects) as shown in the figure below.

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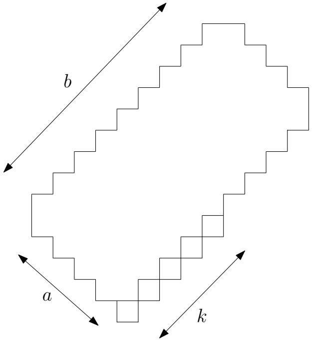

Figure: $\mathcal{AR}_{a,b}^{k}$ with $a = 4, b = 8, k = 4$

General Result

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Theorem

Assume that one of the two shorter sides does not have any defects on it. We assume this to be the southwestern side.

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Assume that one of the two shorter sides does not have any defects on it. We assume this to be the southwestern side. Let $\delta_1, \ldots, \delta_{2n+2k}$ *be the elements of the set* $\{\beta_1, \ldots, \beta_{n+k}\} \cup \{\alpha_1, \ldots, \alpha_n\} \cup \{\gamma_1, \ldots, \gamma_k\}$ *listed in a cyclic order,*

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Then we have

$$\mathrm{M}(\mathcal{AR}_{a,b} \setminus \{\beta_1, \dots, \beta_{n+k}, \alpha_1, \dots, \alpha_n\}) =$$

$$\frac{1}{[\mathrm{M}(\mathcal{AR}^k_{a,b})]^{n-k+1}} \,\mathrm{Pf}[(\mathrm{M}(\mathcal{AR}^k_{a,b} \setminus \{\delta_i, \delta_j\}))_{1 \le i < j \le 2n+2k}],$$

where all the terms on the right hand side are given by explicit formulas.

General Case

General Case

Theorem

Let $\beta_1, \dots, \beta_{n+k}$ *be arbitrary defects of type* β *and* $\alpha_1, \dots, \alpha_n$ *be arbitrary defects of type* α *along the boundary of* $\mathcal{AR}_{a,b}$*. Then* $\mathrm{M}(\mathcal{AR}_{a,b} \setminus \{\beta_1, \dots, \beta_{n+k}, \alpha_1, \dots, \alpha_n\})$ *is equal to the Pfaffian of a* $2n \times 2n$ *matrix whose entries are Pfaffians of* $(2k+2) \times (2k+2)$ *matrices of the type in the statement of main theorem.*

Aztec Rectangles with defects contd.

Aztec Rectangles with defects contd.

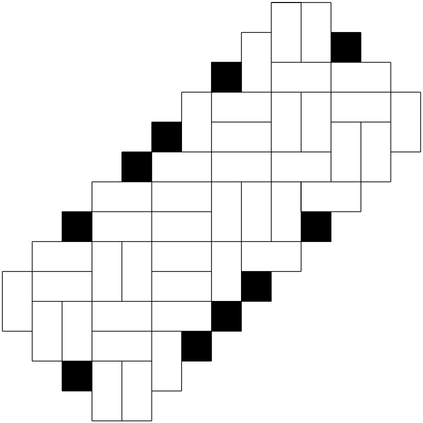

Figure: Tiling with arbitrary defects

Other type of tilings

Other type of tilings

If instead of dominoes, we had trominoes?

Other type of tilings

If instead of dominoes, we had trominoes?

Figure: L-trominoes

Other type of tilings

If instead of dominoes, we had trominoes?

Figure: L-trominoes

The problem becomes quite difficult, and is not solvable using the techniques shown today.

Tromino Cover

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However, such tilings (called covers in this case) exists.

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Theorem

$\mathcal{AR}_{a,b}$ has a cover if and only if $a(b+1)+b(a+1) \equiv 0 \pmod 3$.

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Theorem

A tromino cover for $\mathcal{AR}^k_{a,b}$ can be found in time $O(b^2)$.

Covers with Defects

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With defects the problem becomes even harder.

Covers with Defects

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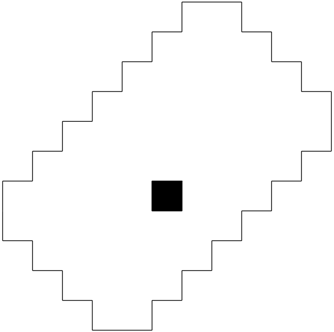

Figure: $\mathcal{AR}_{4,7}$ with defect

Covers with Defects

Covers with Defects

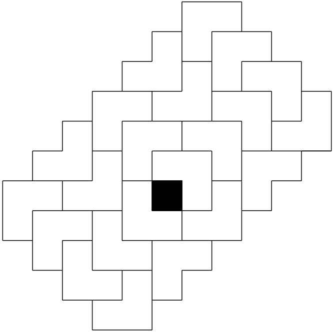

Figure: Covered $\mathcal{AR}_{4,7}$ with defect

Other defects

Other defects

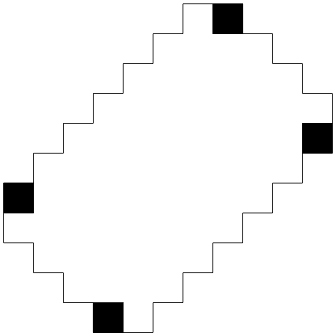

Figure: $\mathcal{AR}_{4,7}$ with defects

Other defects

Other defects

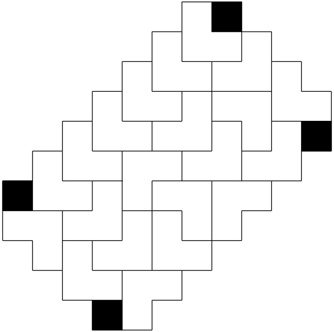

Figure: Covered $\mathcal{AR}_{4,7}$ with defects

Other defects

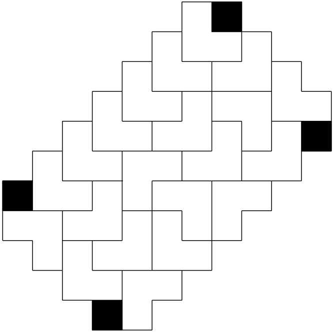

Figure: Covered $\mathcal{AR}_{4,7}$ with defects

Theorem

It is NP-complete to decide if a cover exists for $\mathcal{AR}_{a,b}^{k}$ with fixed number of defects.

Thank you for your attention.