Reading Minds and Other Mathematical Stories

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Reading Minds...

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...with playing cards.

... Other Mathematical Stories

The secret

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- Let's look at an example.

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- For the sake of convinience, and to give a more mathematical feel, let $R = 1$ and $B = 0$ in the sequences of the above type. Phew!

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But, do they even exist for all k?

Nicolaas Govert de Bruijn

Thank you, Wikimedia

Existence

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Form a graph with vertices being the strings of zero-one sequences of length $k - 1$, so there are 2^{k-1} of them. An edge goes from vertex x to vertex y if there is a zero-one string of length k that has x at its left and y at its right.

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Theorem (Euler)

A connected graph has an Eulerian circuit if and only if each vertex has an equal number of edges leading in as leading out.

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This is good news!

Leonhard Euler

The master of us all!

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But, how many of them are there?

Some history...

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Proof? Let's leave it as an exercise?

Is there more time?

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To do another card trick?

Thank you

Thank you, go have some co ee now!