Ranks and Cranks of Partitions

Manjil P. Saikia

University of Vienna

November 16, 2015

Outline

Notations and Preliminaries

For each nonnegative integer n, we set

Notations and Preliminaries

For each nonnegative integer n, we set

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),$$

Notations and Preliminaries

For each nonnegative integer n, we set

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),$$

and

$$(a)_\infty := (a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \mid q \mid < 1.$$

We also set

We also set

$$(a_1, \ldots, a_m; q)_n := (a_1; q)_n \cdots (a_m; q)_n$$

We also set

$$(a_1, \ldots, a_m; q)_n := (a_1; q)_n \cdots (a_m; q)_n$$

and

We also set

$$(a_1, \ldots, a_m; q)_n := (a_1; q)_n \cdots (a_m; q)_n$$

and

$$(a_1, \ldots, a_m; q)_\infty := (a_1; q)_\infty \cdots (a_m; q)_\infty.$$

We also set

$$(a_1, \ldots, a_m; q)_n := (a_1; q)_n \cdots (a_m; q)_n$$

and

$$(a_1, \ldots, a_m; q)_\infty := (a_1; q)_\infty \cdots (a_m; q)_\infty.$$

Ramanujan's general theta function $f(a, b)$ is defined by

We also set

$$(a_1,\dots,a_m;q)_n := (a_1;q)_n\cdots(a_m;q)_n$$

and

$$(a_1,\dots,a_m;q)_\infty := (a_1;q)_\infty\cdots(a_m;q)_\infty.$$

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |\, ab \,| < 1.$$

We also set

$$(a_1, \ldots, a_m; q)_n := (a_1; q)_n \cdots (a_m; q)_n$$

and

$$(a_1, \ldots, a_m; q)_\infty := (a_1; q)_\infty \cdots (a_m; q)_\infty.$$

Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad \mid ab \mid < 1.$$

This satisfies the well-known Jacobi's triple product identity, [6], [11]

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

What is a partition?

Definition (Partition Function)

If n is a positive integer, let $p(n)$ denote the number of unrestricted representations of n as a sum of positive integers, where representations with different orders of the same summands are not regarded as distinct. We call $p(n)$ the partition function.

What is a partition?

Definition (Partition Function)

If n is a positive integer, let $p(n)$ denote the number of unrestricted representations of n as a sum of positive integers, where representations with different orders of the same summands are not regarded as distinct. We call $p(n)$ the partition function.

For example, $p(4) = 5$

What is a partition?

Definition (Partition Function)

If n is a positive integer, let $p(n)$ denote the number of unrestricted representations of n as a sum of positive integers, where representations with different orders of the same summands are not regarded as distinct. We call $p(n)$ the partition function.

For example, $p(4) = 5$, because there are 5 ways to represent 4 as a sum of positive integers,

What is a partition?

Definition (Partition Function)

If n is a positive integer, let $p(n)$ denote the number of unrestricted representations of n as a sum of positive integers, where representations with different orders of the same summands are not regarded as distinct. We call $p(n)$ the partition function.

For example, $p(4) = 5$, because there are 5 ways to represent 4 as a sum of positive integers, namely,
$4 = 3 + 1 = 2 + 2 = 1 + 1 + 1 + 1 = 2 + 1 + 1.$

In 1919, Ramanujan announced that he had found three simple congruences satisfied by $p(n)$ given by

In 1919, Ramanujan announced that he had found three simple congruences satisfied by $p(n)$ given by

$$p(5n+4) \equiv 0 \ (mod \ 5) \tag{1}$$

In 1919, Ramanujan announced that he had found three simple congruences satisfied by $p(n)$ given by

$$p(5n+4) \equiv 0 \ (mod \ 5) \tag{1}$$

$$p(7n+5) \equiv 0 \ (mod \ 7) \tag{2}$$

In 1919, Ramanujan announced that he had found three simple congruences satisfied by $p(n)$ given by

$$p(5n+4) \equiv 0 \ (mod \ 5) \tag{1}$$

$$p(7n+5) \equiv 0 \ (mod \ 7) \tag{2}$$

$$p(11n+6) \equiv 0 \ (mod \ 11) \tag{3}$$

In 1919, Ramanujan announced that he had found three simple congruences satisfied by $p(n)$ given by

$$p(5n+4) \equiv 0 \ (mod\ 5) \tag{1}$$

$$p(7n+5) \equiv 0 \ (mod\ 7) \tag{2}$$

$$p(11n+6) \equiv 0 \ (mod\ 11) \tag{3}$$

Proofs of the above can be found in [6].

Rank of a partition

In order to find combinatorial interpretations of these famous congruences by Ramanujan, in 1944 Freeman Dyson [14] defined the *rank* of a partition to be the largest part minus the number of parts.

Rank of a partition

In order to find combinatorial interpretations of these famous congruences by Ramanujan, in 1944 Freeman Dyson [14] defined the *rank* of a partition to be the largest part minus the number of parts.

Let $N(m, t, n)$ denote the number of partitions of n with rank congruent to m modulo t.

Rank of a partition

In order to find combinatorial interpretations of these famous congruences by Ramanujan, in 1944 Freeman Dyson [14] defined the *rank* of a partition to be the largest part minus the number of parts.

Let $N(m, t, n)$ denote the number of partitions of n with rank congruent to m modulo t. Then Dyson conjectured that

Rank of a partition

In order to find combinatorial interpretations of these famous congruences by Ramanujan, in 1944 Freeman Dyson [14] defined the *rank* of a partition to be the largest part minus the number of parts.

Let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. Then Dyson conjectured that

$$N(k,5,5n+4)=\frac{p(5n+4)}{5},\quad 0\leq k\leq 4,$$

and

$$N(k,7,7n+5)=\frac{p(7n+5)}{7},\quad 0\leq k\leq 6,$$

which yield combinatorial interpretations of (1) and (2).

Rank of a partition

These conjectures were later proven by Atkin and Swinnerton-Dyer.

Rank of a partition

These conjectures were later proven by Atkin and Swinnerton-Dyer. The generating function for $N(m, n)$ is given by

Rank of a partition

These conjectures were later proven by Atkin and Swinnerton-Dyer. The generating function for $N(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N(m,n)a^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(aq;q)_n(q/a;q)_n}$$

Here $\mid q \mid < 1, \quad \mid q \mid < \mid a \mid < 1/ \mid q \mid$.

Rank of a partition

These conjectures were later proven by Atkin and Swinnerton-Dyer. The generating function for $N(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N(m,n)a^m q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(aq;q)_n(q/a;q)_n}$$

Here $\mid q \mid < 1, \quad \mid q \mid < \mid a \mid < 1/\mid q \mid$.

However, the corresponding analogue of the rank doesn't hold for (3), and so Dyson conjectured the existence of another statistic which he called *crank*.

Outline

Vector Crank

In his doctoral dissertation F. G. Garvan defined vector partitions which became the forerunner of the true crank.

Vector Crank

In his doctoral dissertation F. G. Garvan defined vector partitions which became the forerunner of the true crank.
Let $\mathcal{P}$ denote the set of partitions and $\mathcal{D}$ denote the set of partitions into distinct parts. Following Garvan, [16] we denote the set of vector partitions V to be defined by

$$V = \mathcal{D} \times \mathcal{P} \times \mathcal{P}.$$

Vector Crank

In his doctoral dissertation F. G. Garvan defined vector partitions which became the forerunner of the true crank.
Let $\mathcal{P}$ denote the set of partitions and $\mathcal{D}$ denote the set of partitions into distinct parts. Following Garvan, [16] we denote the set of vector partitions V to be defined by

$$V = \mathcal{D} \times \mathcal{P} \times \mathcal{P}.$$

For $\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V$, we define the weight $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)}$, the crank$(\vec{\pi}) = \#(\pi_2) - \#(\pi_3)$, and $\mid \vec{\pi} \mid = \mid \pi_1 \mid + \mid \pi_2 \mid + \mid \pi_3 \mid$, where $\mid \pi \mid$ is the sum of the parts of π and $\#(\pi_i)$ is the number of parts of π_i.

Vector crank

The number of vector partitions of n with crank m counted according to the weight ω is denoted by

$$N_V(m,n) = \sum_{\vec{\pi}\in V, |\vec{\pi}|=n, crank(\vec{\pi})=m} \omega(\vec{\pi}).$$

Vector crank

The number of vector partitions of n with crank m counted according to the weight ω is denoted by

$$N_V(m,n) = \sum_{\vec{\pi}\in V, |\vec{\pi}|=n, crank(\vec{\pi})=m} \omega(\vec{\pi}).$$

We then have

$$\sum_m N_V(m,n) = p(n).$$

Vector crank

Let $N_V(m, t, n)$ denote the number of vector partitions of n with crank congruent to m modulo t counted according to the weights ω. Then we have

Vector crank

Let $N_V(m, t, n)$ denote the number of vector partitions of n with crank congruent to m modulo t counted according to the weights ω. Then we have

Theorem (Garvan, [16])

$$N_V(k, 5, 5n+4) = \frac{p(5n+4)}{5}, \ 0 \leq k \leq 4,$$

$$N_V(k, 7, 7n+5) = \frac{p(7n+5)}{7}, \ 0 \leq k \leq 6,$$

$$N_V(k, 11, 11n+6) = \frac{p(11n+6)}{11}, \ 0 \leq k \leq 10.$$

True Crank

On June 6, 1987 at a student dormitory at the University of Illinois, G. E. Andrews and F. G. Garvan [5] found the true crank.

True Crank

On June 6, 1987 at a student dormitory at the University of Illinois, G. E. Andrews and F. G. Garvan [5] found the true crank.

Definition (Crank)

For a partition π, let $\lambda(\pi)$ denote the largest part of π, let $\mu(\pi)$ denote the number of ones in π, and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is then defined to be

True Crank

On June 6, 1987 at a student dormitory at the University of Illinois, G. E. Andrews and F. G. Garvan [5] found the true crank.

Definition (Crank)

For a partition π, let $\lambda(\pi)$ denote the largest part of π, let $\mu(\pi)$ denote the number of ones in π, and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is then defined to be

$$c(\pi) = \begin{cases} \lambda(\pi) & \textit{if } \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi) & \textit{if } \mu(\pi) > 0. \end{cases}$$

Crank of a partition

Let $M(m, n)$ denote the number of partitions of n with crank m, and let $M(m, t, n)$ denote the number of partitions of n with crank congruent to m modulo t.

Crank of a partition

Let $M(m, n)$ denote the number of partitions of n with crank m, and let $M(m, t, n)$ denote the number of partitions of n with crank congruent to m modulo t.
For $n \leq 1$ we set $M(0, 0) = 1, M(m, 0) = 0$, otherwise $M(0, 1) = -1, M(1, 1) = M(-1, 1) = 1$ and $M(m, 1) = 0$ otherwise.

Crank of a partition

Let $M(m, n)$ denote the number of partitions of n with crank m, and let $M(m, t, n)$ denote the number of partitions of n with crank congruent to m modulo t.

For $n \leq 1$ we set $M(0, 0) = 1, M(m, 0) = 0$, otherwise $M(0, 1) = -1, M(1, 1) = M(-1, 1) = 1$ and $M(m, 1) = 0$ otherwise.

The generating function for $M(m, n)$ is given by

Crank of a partition

Let $M(m,n)$ denote the number of partitions of n with crank m, and let $M(m,t,n)$ denote the number of partitions of n with crank congruent to m modulo t.
For $n \leq 1$ we set $M(0,0)=1, M(m,0)=0$, otherwise $M(0,1)=-1, M(1,1)=M(-1,1)=1$ and $M(m,1)=0$ otherwise.
The generating function for $M(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)a^m q^n = \frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty}. \tag{4}$$

Crank of a partition

Let $M(m, n)$ denote the number of partitions of n with crank m, and let $M(m, t, n)$ denote the number of partitions of n with crank congruent to m modulo t.

For $n \leq 1$ we set $M(0, 0) = 1, M(m, 0) = 0$, otherwise $M(0, 1) = -1, M(1, 1) = M(-1, 1) = 1$ and $M(m, 1) = 0$ otherwise.

The generating function for $M(m, n)$ is given by

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m, n) a^m q^n = \frac{(q; q)_\infty}{(aq; q)_\infty (q/a; q)_\infty}. \qquad (4)$$

The crank not only leads to combinatorial interpretations of (1) and (2), but also of (3).

Crank of a partition

Theorem (Andrews-Gravan [5])

With $M(m, t, n)$ defined as above,

Crank of a partition

Theorem (Andrews-Gravan [5])

With $M(m, t, n)$ defined as above,

$$M(k, 5, 5n+4) = \frac{p(5n+4)}{5}, \ 0 \leq k \leq 4,$$

$$M(k, 7, 7n+5) = \frac{p(7n+5)}{7}, \ 0 \leq k \leq 6,$$

$$M(k, 11, 11n+6) = \frac{p(11n+6)}{11}, \ 0 \leq k \leq 10.$$

Crank of a partition

Theorem (Andrews-Gravan [5])

With $M(m, t, n)$ defined as above,

$$M(k, 5, 5n+4) = \frac{p(5n+4)}{5}, \ 0 \leq k \leq 4,$$

$$M(k, 7, 7n+5) = \frac{p(7n+5)}{7}, \ 0 \leq k \leq 6,$$

$$M(k, 11, 11n+6) = \frac{p(11n+6)}{11}, \ 0 \leq k \leq 10.$$

An excellent introduction to cranks is given by Garvan in [17].

Ramanujan and cranks

In page 179 of [19], we find the generating function for cranks (4) in the form

$$F(q) := F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} =: \sum_{n=0}^{\infty} \lambda_n q^n.$$

Ramanujan and cranks

In page 179 of [19], we find the generating function for cranks (4) in the form

$$F(q) := F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} =: \sum_{n=0}^{\infty} \lambda_n q^n.$$

And so from (4), we have

$$\lambda_n = \sum_{m=-\infty}^{\infty} M(m,n)a^m.$$

Ramanujan and cranks

In page 179 of [19], we find the generating function for cranks (4) in the form

$$F(q) := F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} =: \sum_{n=0}^{\infty} \lambda_n q^n.$$

And so from (4), we have

$$\lambda_n = \sum_{m=-\infty}^{\infty} M(m,n) a^m.$$

We note that if $a = 1$ in the above, then it reduces to the generating function of $p(n)$.

Dissections

Definition (Dissections)

If

$$P(q) := \sum_{n=0}^{\infty} a_n q^n$$

is any power series, then the m-dissection of $P(q)$ is given by

$$P(q) = \sum_{j=0}^{m-1} \sum_{n_j=0}^{\infty} a_{n_j m+j} q^{n_j m+j}.$$

Dissections

Definition (Dissections)

If

$$P(q) := \sum_{n=0}^{\infty} a_n q^n$$

is any power series, then the m-dissection of $P(q)$ is given by

$$P(q) = \sum_{j=0}^{m-1} \sum_{n_j=0}^{\infty} a_{n_j m+j} q^{n_j m+j}.$$

In his lost notebook [19] Ramanujan offers, in various guises, m-dissections for $F_a(q)$ for $m = 2, 3, 5, 7, 11$.

Dissections

Definition (Dissections)

If

$$P(q) := \sum_{n=0}^{\infty} a_n q^n$$

is any power series, then the m-dissection of $P(q)$ is given by

$$P(q) = \sum_{j=0}^{m-1} \sum_{n_j=0}^{\infty} a_{n_j m+j} q^{n_j m+j}.$$

In his lost notebook [19] Ramanujan offers, in various guises, m-dissections for $F_a(q)$ for $m = 2, 3, 5, 7, 11$. In particular pon page 179 Ramanujan offers 2- and 3- dissections for $F_q(q)$ in the form of congruences.

2-dissection

Theorem (2-dissection)

$$F_a(\sqrt{q}) \equiv \frac{f(-q^3;-q^5)}{(-q^2;q^2)_\infty} + (a-1+\frac{1}{a})\sqrt{q}\frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \pmod{a^2+\frac{1}{a^2}}. \quad (5)$$

2-dissection

Theorem (2-dissection)

$$F_a(\sqrt{q}) \equiv \frac{f(-q^3;-q^5)}{(-q^2;q^2)_\infty} + (a-1+\frac{1}{a})\sqrt{q}\frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \ (mod\ a^2+\frac{1}{a^2}). \tag{5}$$

We note that $\lambda_2 = a^2 + a^{-2}$, which trivially implies that $a^4 \equiv -1 \ (mod\ \lambda_2)$ and $a^8 \equiv 1 \ (mod\ \lambda_2)$. Thus, in (5) a behaves like a primitive 8th root of unity modulo λ_2.

2-dissection

Theorem (2-dissection)

$$F_a(\sqrt{q}) \equiv \frac{f(-q^3;-q^5)}{(-q^2;q^2)_\infty} + (a-1+\frac{1}{a})\sqrt{q}\frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \ (mod\ a^2+\frac{1}{a^2}). \tag{5}$$

We note that $\lambda_2 = a^2 + a^{-2}$, which trivially implies that $a^4 \equiv -1 \ (mod\ \lambda_2)$ and $a^8 \equiv 1 \ (mod\ \lambda_2)$. Thus, in (5) a behaves like a primitive 8th root of unity modulo λ_2.

Thus, if we let $a = exp(2\pi i/8)$ and replace q by q^2 in the definition of a dissection, (5) will give the 2-dissection of $F_a(q)$.

3-dissection

Theorem (3-dissection)

$$F_q(q^{1/3}) \equiv \frac{f(-q^2,-q^7)f(-q^4,-q^5)}{(q^9;q^9)_\infty}$$

$$+(a\text{-}1+1/a)q^{1/3}\frac{f(-q,-q^8)f(-q^4,-q^5)}{(q^9;q^9)_\infty}$$

$$+(a^2+\frac{1}{a^2})q^{2/3}\frac{f(-q,-q^8)f(-q^2,-q^7)}{(q^9;q^9)_\infty} \quad (mod\ a^3+1+\frac{1}{a^3}).(6)$$

3-dissection

Theorem (3-dissection)

$$F_q(q^{1/3}) \equiv \frac{f(-q^2,-q^7)f(-q^4,-q^5)}{(q^9;q^9)_\infty}$$

$$+(a\text{-}1+1/a)q^{1/3}\frac{f(-q,-q^8)f(-q^4,-q^5)}{(q^9;q^9)_\infty}$$
$$+(a^2+\frac{1}{a^2})q^{2/3}\frac{f(-q,-q^8)f(-q^2,-q^7)}{(q^9;q^9)_\infty}\ (mod\ a^3+1+\frac{1}{a^3}).(6)$$

Again we note that, $\lambda_3 = a^3 + 1 + \frac{1}{a^3}$, from which it follows that $a^9 \equiv -a^6 - a^3 \equiv 1\ (mod\ \lambda_3)$. So in (6), a behaves like a primitive 9th root of unity.

3-dissection

Theorem (3-dissection)

$$F_q(q^{1/3}) \equiv \frac{f(-q^2,-q^7)f(-q^4,-q^5)}{(q^9;q^9)_\infty}$$

$$+(a\text{-}1+1/a)q^{1/3}\frac{f(-q,-q^8)f(-q^4,-q^5)}{(q^9;q^9)_\infty}$$
$$+(a^2+\frac{1}{a^2})q^{2/3}\frac{f(-q,-q^8)f(-q^2,-q^7)}{(q^9;q^9)_\infty} \ (mod\ a^3+1+\frac{1}{a^3}).(6)$$

Again we note that, $\lambda_3 = a^3+1+\frac{1}{a^3}$, from which it follows that $a^9 \equiv -a^6 - a^3 \equiv 1\ (mod\ \lambda_3)$. So in (6), a behaves like a primitive 9th root of unity.

While if we let $a = exp(2\pi i/9)$ and replace q by q^3 in the definition of a dissection, (6) will give the 3-dissection of $F_a(q)$.

5-dissection

In contrast to (5) and (6), Ramanujan offererd the 5-dissection in terms of an equality.

5-dissection

In contrast to (5) and (6), Ramanujan offererd the 5-dissection in terms of an equality.

Theorem (5-dissection)

$$F_a(q) = \frac{f(-q^2,-q^3)}{f^2(-q,-q^4)}f^2(-q^5)-$$

$$4\cos^2(2n\pi/5)q^{1/5}\frac{f^2(-q^5)}{f(-q,-q^4)} + 2\cos(4n\pi/5)q^{2/5}\frac{f^2(-q^5)}{f(-q^2,-q^3)}$$
$$-2\cos(2n\pi/5)q^{3/5}\frac{f(-q,-q^4)}{f^2(-q^2,-q^3)}f^2(-q^5).(7)$$

5-dissection

In contrast to (5) and (6), Ramanujan offererd the 5-dissection in terms of an equality.

Theorem (5-dissection)

$$F_a(q) = \frac{f(-q^2,-q^3)}{f^2(-q,-q^4)}f^2(-q^5) -$$

$$4\cos^2(2n\pi/5)q^{1/5}\frac{f^2(-q^5)}{f(-q,-q^4)} + 2\cos(4n\pi/5)q^{2/5}\frac{f^2(-q^5)}{f(-q^2,-q^3)}$$

$$-2\cos(2n\pi/5)q^{3/5}\frac{f(-q,-q^4)}{f^2(-q^2,-q^3)}f^2(-q^5).(7)$$

We observe that (7) has no term with $q^{4/5}$, which is a reflection of (1).

5-dissection

In contrast to (5) and (6), Ramanujan offererd the 5-dissection in terms of an equality.

Theorem (5-dissection)

$$F_a(q) = \frac{f(-q^2,-q^3)}{f^2(-q,-q^4)} f^2(-q^5) -$$

$4\cos^2(2n\pi/5)q^{1/5}\frac{f^2(-q^5)}{f(-q,-q^4)} + 2\cos(4n\pi/5)q^{2/5}\frac{f^2(-q^5)}{f(-q^2,-q^3)}$
-$2\cos(2n\pi/5)q^{3/5}\frac{f(-q,-q^4)}{f^2(-q^2,-q^3)}f^2(-q^5)$.(7)

We observe that (7) has no term with $q^{4/5}$, which is a reflection of (1).
In fact, one can replace (7) by a congruence and in turn (5) and (6) by equalities. This is done in [8].

Other dissections

Ramanujan did not specifically give the 7- and 11-dissections of $F_a(q)$ in [19].

Other dissections

Ramanujan did not specifically give the 7- and 11-dissections of $F_a(q)$ in [19]. However, he vaugely gives some of the coefficients occuring in those dissections.

Other dissections

Ramanujan did not specifically give the 7- and 11-dissections of $F_a(q)$ in [19]. However, he vaugely gives some of the coefficients occuring in those dissections.

Uniform proofs of these dissections and the others already stated earlier are given in [8].

Other results

In [19], Ramanujan also recorded various other results related to cranks.

Other results

In [19], Ramanujan also recorded various other results related to cranks.
For example in pages 179 and 180, he gave tables of n for which λ_n satisfies certain congruences.

Other results

In [19], Ramanujan also recorded various other results related to cranks.
For example in pages 179 and 180, he gave tables of n for which λ_n satisfies certain congruences. All these claims are studied systematically in [2, 9] and [10].

Other results

In [19], Ramanujan also recorded various other results related to cranks.
For example in pages 179 and 180, he gave tables of n for which λ_n satisfies certain congruences. All these claims are studied systematically in [2, 9] and [10].
Ramanujan also recorded the first 21 coefficients in the power series of $F_a(q)$ in page 58.

The final problem

G. N. Watson in his famous presidential address of the London Mathematical Society mentioned that mock theta functions was the final mathematical object that Ramanujan worked on.

The final problem

G. N. Watson in his famous presidential address of the London Mathematical Society mentioned that mock theta functions was the final mathematical object that Ramanujan worked on. This observation was based on the fact that Ramanujan wrote to G. H. Hardy in January 1920, just three months before his death about mock theta functions.

The final problem

G. N. Watson in his famous presidential address of the London Mathematical Society mentioned that mock theta functions was the final mathematical object that Ramanujan worked on. This observation was based on the fact that Ramanujan wrote to G. H. Hardy in January 1920, just three months before his death about mock theta functions.

However, in recent years [10] it has come to light that Ramanujan kept two stacks of papers on which he worked, one of which was for scratch work.

The final problem

G. N. Watson in his famous presidential address of the London Mathematical Society mentioned that mock theta functions was the final mathematical object that Ramanujan worked on. This observation was based on the fact that Ramanujan wrote to G. H. Hardy in January 1920, just three months before his death about mock theta functions.

However, in recent years [10] it has come to light that Ramanujan kept two stacks of papers on which he worked, one of which was for scratch work. After scrutiny, it has been found that Ramanujan did sctarch work related to cranks in his final days, that is around three days before his death.

The final problem

G. N. Watson in his famous presidential address of the London Mathematical Society mentioned that mock theta functions was the final mathematical object that Ramanujan worked on. This observation was based on the fact that Ramanujan wrote to G. H. Hardy in January 1920, just three months before his death about mock theta functions.

However, in recent years [10] it has come to light that Ramanujan kept two stacks of papers on which he worked, one of which was for scratch work. After scrutiny, it has been found that Ramanujan did sctarch work related to cranks in his final days, that is around three days before his death.

It is therefore, very likely that the final problem on which Ramanujan worked was cranks, although it is very unlikely that Ramanujan thought about the combinatorial aspects of cranks.

Generalizations of Ramanujan's Partition Congrunces

The Ramanujan partition functions (1), (2) and (3) have been a long standing subject of study.

Generalizations of Ramanujan's Partition Congrunces

The Ramanujan partition functions (1), (2) and (3) have been a long standing subject of study.
Various people have tried to find generalizations of these beautiful congruences.

Generalizations of Ramanujan's Partition Congrunces

The Ramanujan partition functions (1), (2) and (3) have been a long standing subject of study.

Various people have tried to find generalizations of these beautiful congruences.

Recently, the work of Ken Ono has revealed that similar congruences for $p(n)$ exists for any modulus coprime to 6.

Generalizations of Ramanujan's Partition Congrunces

The Ramanujan partition functions (1), (2) and (3) have been a long standing subject of study.

Various people have tried to find generalizations of these beautiful congruences.

Recently, the work of Ken Ono has revealed that similar congruences for $p(n)$ exists for any modulus coprime to 6.But this leaves open the characterizations of those values of n for which $p(n)$ is divisible by moduli not coprime to 6, in particular moduli 2 and 3.

Generalizations of Ramanujan's Partition Congrunces

The Ramanujan partition functions (1), (2) and (3) have been a long standing subject of study.
Various people have tried to find generalizations of these beautiful congruences.
Recently, the work of Ken Ono has revealed that similar congruences for $p(n)$ exists for any modulus coprime to 6.But this leaves open the characterizations of those values of n for which $p(n)$ is divisible by moduli not coprime to 6, in particular moduli 2 and 3.
Recent work by Kaavya [18] has shared some light on new means of attacking the problem for determining when $p(n) \equiv 0 \bmod 2$.

Crank 0 partitions and $p(n)$

It was shown by Berkovich and Garvan that there is always an involution between the crank k and crank $-k$ partitions of n for all values of k.

Crank 0 partitions and $p(n)$

It was shown by Berkovich and Garvan that there is always an involution between the crank k and crank $-k$ partitions of n for all values of k.
Thus

$$p_k(n) = p_{-k}(n).$$

Crank 0 partitions and $p(n)$

It was shown by Berkovich and Garvan that there is always an involution between the crank k and crank $-k$ partitions of n for all values of k.
Thus

$$p_k(n) = p_{-k}(n).$$

Hence it follows that the crank 0 partitions of n have the same parity as $p(n)$ itself.

Crank 0 partitions and $p(n)$

It was shown by Berkovich and Garvan that there is always an involution between the crank k and crank $-k$ partitions of n for all values of k.
Thus

$$p_k(n) = p_{-k}(n).$$

Hence it follows that the crank 0 partitions of n have the same parity as $p(n)$ itself.Since such partitions are less in number than $p(n)$ so it may be eaier to investigate the parity of $p_0(n)$ than $p(n)$.

Crank 0 partitions and $p(n)$

It was shown by Berkovich and Garvan that there is always an involution between the crank k and crank $-k$ partitions of n for all values of k.
Thus

$$p_k(n) = p_{-k}(n).$$

Hence it follows that the crank 0 partitions of n have the same parity as $p(n)$ itself.Since such partitions are less in number than $p(n)$ so it may be eaier to investigate the parity of $p_0(n)$ than $p(n)$.
Now the question comes how to make a crank 0 partition?

Ferrers Diagram

A crank 0 partition is made by building a Ferrers diagram on something called a k-root.

Ferrers Diagram

A crank 0 partition is made by building a Ferrers diagram on something called a k-root. This is a k by $k+1$ rectangle of dots, with a column of k dots underneath, for a total of k^2+2k dots.

Ferrers Diagram

A crank 0 partition is made by building a Ferrers diagram on something called a k-root. This is a k by $k+1$ rectangle of dots, with a column of k dots underneath, for a total of $k^2 + 2k$ dots. We observe that this is the smallest number for which there is a crank 0 partition with exactly k parts equal to 1.

Ferrers Diagram

A crank 0 partition is made by building a Ferrers diagram on something called a k-root.This is a k by $k+1$ rectangle of dots, with a column of k dots underneath, for a total of k^2+2k dots. We observe that this is the smallest number for which there is a crank 0 partition with exactly k parts equal to 1. We call the k by $k+1$ rectangle the *body* and the column of k dots the *tail*.

Ferrers Diagram

A crank 0 partition is made by building a Ferrers diagram on something called a k-root.This is a k by $k+1$ rectangle of dots, with a column of k dots underneath, for a total of k^2+2k dots. We observe that this is the smallest number for which there is a crank 0 partition with exactly k parts equal to 1. We call the k by $k+1$ rectangle the *body* and the column of k dots the *tail*. The Ferrers diagram of crank 0 partitions can be built as follows, first in between the body and the tail new rows of length greater than 1 and less than $k+1$ may be inserted in order of non-increasing length.

Ferrers Diagram

A crank 0 partition is made by building a Ferrers diagram on something called a k-root.This is a k by $k+1$ rectangle of dots, with a column of k dots underneath, for a total of k^2+2k dots. We observe that this is the smallest number for which there is a crank 0 partition with exactly k parts equal to 1. We call the k by $k+1$ rectangle the *body* and the column of k dots the *tail*. The Ferrers diagram of crank 0 partitions can be built as follows, first in between the body and the tail new rows of length greater than 1 and less than $k+1$ may be inserted in order of non-increasing length. Second, to the right of the body any number of columns of length k may be inserted again in order of non-decreasing length.

Generating function of $p_0(n)$

We denote the number of crank m partitions of n which are built on k-roots as $p_m^k(n)$.

Generating function of $p_0(n)$

We denote the number of crank m partitions of n which are built on k-roots as $p_m^k(n)$. This is also the number of partitions of n such that the crank equals m and the number of parts equal to 1 is exactly k.

Generating function of $p_0(n)$

We denote the number of crank m partitions of n which are built on k-roots as $p_m^k(n)$. This is also the number of partitions of n such that the crank equals m and the number of parts equal to 1 is exactly k.
We have

$$p_0(n) = p_0^1(n) + p_0^2(n) + p_0^3(n) + \cdots .$$

Generating function of $p_0(n)$

We denote the number of crank m partitions of n which are built on k-roots as $p_m^k(n)$. This is also the number of partitions of n such that the crank equals m and the number of parts equal to 1 is exactly k.

We have

$$p_0(n) = p_0^1(n) + p_0^2(n) + p_0^3(n) + \cdots .$$

For each k we note that the terms in the generating function of $p_0^k(n)$ has one-one correspondence with the Ferrers diagram for crank 0 partitions built up from the k-root.

In fact the technique is used to get the following

In fact the technique is used to get the following

Theorem (Kaavya [18])

The generating function for $p_0(n)$ is given by

$$\sum_{n=1}^{\infty} p_0(n)q^n = (1-q)\sum_{k=1}^{\infty} \frac{q^{k^2+2k}}{(q;q)_k^2}.$$

Outline

Weighted partitions

In the study of partition identities, it has often been of interest to consider weighted counts of partitions.

Weighted partitions

In the study of partition identities, it has often been of interest to consider weighted counts of partitions.
For example we have the following beautiful result

Weighted partitions

In the study of partition identities, it has often been of interest to consider weighted counts of partitions.
For example we have the following beautiful result

Theorem (Fokkink, Fokkink and Wang, [15])

If $\mathcal{D}_n$ denotes the set of partitions π of n into distinct parts, then

$$-\sum_{\pi\in\mathcal{D}_n}(-1)^{\#(\pi)}\sigma(\pi)=d(n),$$

where $\#(\pi)$ is the number of parts of π, $\sigma(\pi)$ is the smallest part of π and $d(n)$ is the number of divisors of n.

The spt-function

We denote by $spt(n)$ the total number of appearences of the smallest part in each integer partition of n.

The spt-function

We denote by $spt(n)$ the total number of appearences of the smallest part in each integer partition of n. For example, from the previous example we have $spt(4) = 10$.

In [1], G. E. Andrews found some remarkable properties of the spt-function which reminds us of the Ramanujan congruence of partitions.

The spt-function

We denote by $spt(n)$ the total number of appearences of the smallest part in each integer partition of n. For example, from the previous example we have $spt(4) = 10$.
In [1], G. E. Andrews found some remarkable properties of the spt-function which reminds us of the Ramanujan congruence of partitions.

$$spt(5n+4) \equiv 0 \ (mod\ 5) \tag{8}$$

The spt-function

We denote by $spt(n)$ the total number of appearences of the smallest part in each integer partition of n. For example, from the previous example we have $spt(4) = 10$.
In [1], G. E. Andrews found some remarkable properties of the spt-function which reminds us of the Ramanujan congruence of partitions.

$$spt(5n+4) \equiv 0 \ (mod\ 5) \tag{8}$$

$$spt(7n+5) \equiv 0 \ (mod\ 7) \tag{9}$$

The spt-function

We denote by $spt(n)$ the total number of appearences of the smallest part in each integer partition of n. For example, from the previous example we have $spt(4) = 10$.
In [1], G. E. Andrews found some remarkable properties of the spt-function which reminds us of the Ramanujan congruence of partitions.

$$spt(5n+4) \equiv 0 \ (mod\ 5) \tag{8}$$

$$spt(7n+5) \equiv 0 \ (mod\ 7) \tag{9}$$

$$spt(13n+6) \equiv 0 \ (mod\ 13) \tag{10}$$

Generating function for the spt-function

In [1], Andrews also found the generating function for the spt-function.

Generating function for the spt-function

In [1], Andrews also found the generating function for the spt-function.

Theorem

$$\sum_{n\geq 1} spt(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$$
$$+\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2}.$$

spt-Crank

Like the crank for the ordinary partition function, it was believed that there exists a crank for the spt-function, in order to explain (8), (9) and (10).

spt-Crank

Like the crank for the ordinary partition function, it was believed that there exists a crank for the spt-function, in order to explain (8), (9) and (10).

In [4], Andrews, Garvan and L. Liang found a vector crank, like the one found by Garvan which explained the first two congruences for the spt-function.

spt-Crank

Like the crank for the ordinary partition function, it was believed that there exists a crank for the spt-function, in order to explain (8), (9) and (10).

In [4], Andrews, Garvan and L. Liang found a vector crank, like the one found by Garvan which explained the first two congruences for the spt-function.

For a partition π, we define by $s(\pi)$, the smallest part that appears in π. For the empty partition, we have $s(-) = \infty$.

spt-Crank

Like the crank for the ordinary partition function, it was believed that there exists a crank for the spt-function, in order to explain (8), (9) and (10).

In [4], Andrews, Garvan and L. Liang found a vector crank, like the one found by Garvan which explained the first two congruences for the spt-function.

For a partition π, we define by $s(\pi)$, the smallest part that appears in π. For the empty partition, we have $s(-) = \infty$. We define the following subset of the set of vector partitions

spt-Crank

Like the crank for the ordinary partition function, it was believed that there exists a crank for the spt-function, in order to explain (8), (9) and (10).

In [4], Andrews, Garvan and L. Liang found a vector crank, like the one found by Garvan which explained the first two congruences for the spt-function.

For a partition π, we define by $s(\pi)$, the smallest part that appears in π. For the empty partition, we have $s(-) = \infty$. We define the following subset of the set of vector partitions

$$S := \{\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V : 1 \leq s(\pi_1) < \infty \, and \, s(\pi_1) \leq \min(s(\pi_2), s(\pi_3$$

spt-Crank

Like the crank for the ordinary partition function, it was believed that there exists a crank for the spt-function, in order to explain (8), (9) and (10).

In [4], Andrews, Garvan and L. Liang found a vector crank, like the one found by Garvan which explained the first two congruences for the spt-function.

For a partition π, we define by $s(\pi)$, the smallest part that appears in π. For the empty partition, we have $s(-) = \infty$. We define the following subset of the set of vector partitions

$$S := \{\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V : 1 \leq s(\pi_1) < \infty \textit{ and } s(\pi_1) \leq \min(s(\pi_2), s(\pi_3$$

For $\vec{\pi} \in S$, we define the weight ω_1 by $\omega_1(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$. The number of vector partitions of n in S with crank m counted according to the weight ω_1 is denoted by $N_S(m, n)$.

spt-Crank

We can prove that $N_S(m,n) = N_S(-m,n)$ and using a similar notation we have $N_S(m,t,n) = N_S(t-m,t,n)$.

spt-Crank

We can prove that $N_S(m,n) = N_S(-m,n)$ and using a similar notation we have $N_S(m,t,n) = N_S(t-m,t,n)$.
The main result in [4] is the following

spt-Crank

We can prove that $N_S(m,n) = N_S(-m,n)$ and using a similar notation we have $N_S(m,t,n) = N_S(t-m,t,n)$.
The main result in [4] is the following

Theorem (Andrews, Garvan and Liang, [4])

$$N_S(k,5,5n+4) = \frac{spt(5n+4)}{5}, \ 0 \leq k \leq 4,$$

$$N_S(k,7,7n+5) = \frac{spt(7n+5)}{7}, \ 0 \leq k \leq 6.$$

spt-Crank

We can prove that $N_S(m, n) = N_S(-m, n)$ and using a similar notation we have $N_S(m, t, n) = N_S(t - m, t, n)$.
The main result in [4] is the following

Theorem (Andrews, Garvan and Liang, [4])

$$N_S(k, 5, 5n + 4) = \frac{spt(5n + 4)}{5}, \ 0 \le k \le 4,$$

$$N_S(k, 7, 7n + 5) = \frac{spt(7n + 5)}{7}, \ 0 \le k \le 6.$$

This explained the first two congruences of the spt-function. It remains an open question to find a result like the above for the third congruence.

spt-Crank

In [17], it was proven that $N_V(m, n) \geq 0$.

spt-Crank

In [17], it was proven that $N_V(m,n) \geq 0$.
We also have the following remarkable result

Theorem (Andrews, Garvan and Liang, [4])

$$N_S(m,n) \geq 0$$

for all (m,n).

Andrews-Dyson-Rhoades Conjecture

In a recent paper, [3] Andrews, Dyson and R. C. Rhoades have given the following conjecture on the basis of numerical data.

Andrews-Dyson-Rhoades Conjecture

In a recent paper, [3] Andrews, Dyson and R. C. Rhoades have given the following conjecture on the basis of numerical data.

Conjecture

Andrews-Dyson-Rhoades Conjecture For each $m \geq 0$ *and* $n \geq 0$ *we have,* $N_S(m, n) \geq N_S(m + 1, n)$.

Andrews-Dyson-Rhoades Conjecture

In a recent paper, [3] Andrews, Dyson and R. C. Rhoades have given the following conjecture on the basis of numerical data.

Conjecture

Andrews-Dyson-Rhoades Conjecture For each $m \geq 0$ *and* $n \geq 0$ *we have,* $N_S(m, n) \geq N_S(m+1, n)$.

W. Y. C. Chen, K. Q. Ji and W. J. T. Zang has recently announced a proof of this conjecture, [13].

Marked Partitions

In [3], the authors defined what are called marked partitions.

Marked Partitions

In [3], the authors defined what are called marked partitions.

Definition (Marked Partition)

A marked partition is a pair (λ, k) where λ is a partition and k is an integer identifying one of its smallest parts. If there are s smallest parts then $k = 1, 2, \ldots, s$.

Marked Partitions

In [3], the authors defined what are called marked partitions.

Definition (Marked Partition)

A marked partition is a pair (λ, k) where λ is a partition and k is an integer identifying one of its smallest parts. If there are s smallest parts then $k = 1, 2, \ldots, s$.

They also proposed a challenge to define the spt-Crank in terms of ordinary partitions or marked partitions.

Marked Partitions

In [3], the authors defined what are called marked partitions.

Definition (Marked Partition)

A marked partition is a pair (λ, k) where λ is a partition and k is an integer identifying one of its smallest parts. If there are s smallest parts then $k = 1, 2, \ldots, s$.

They also proposed a challenge to define the spt-Crank in terms of ordinary partitions or marked partitions.
This challenge was accepted by Chen, Qi and Zang [12] and they found a neat combinatorial way to define the crank in terms of marked and doubly marked partitions.

Doubly Marked Partitions

Definition (Doubly Marked Partition)

A doubly marked partition of n is an ordinary partition λ of n along with two distinguished columns indexed by s and t, denoted (λ, s, t) where

Doubly Marked Partitions

Definition (Doubly Marked Partition)

A doubly marked partition of n is an ordinary partition λ of n along with two distinguished columns indexed by s and t, denoted (λ, s, t) where

1. $1 \leq s \leq D(\lambda)$,

Doubly Marked Partitions

Definition (Doubly Marked Partition)

A doubly marked partition of n is an ordinary partition λ of n along with two distinguished columns indexed by s and t, denoted (λ, s, t) where

1. $1 \leq s \leq D(\lambda)$,
2. $s \leq t \leq \lambda_t$,

Doubly Marked Partitions

Definition (Doubly Marked Partition)

A doubly marked partition of n is an ordinary partition λ of n along with two distinguished columns indexed by s and t, denoted (λ, s, t) where

1. $1 \leq s \leq D(\lambda)$,
2. $s \leq t \leq \lambda_t$,
3. $\lambda'_s = \lambda'_t$.

Doubly Marked Partitions

Definition (Doubly Marked Partition)

A doubly marked partition of n is an ordinary partition λ of n along with two distinguished columns indexed by s and t, denoted (λ, s, t) where

1. $1 \leq s \leq D(\lambda)$,
2. $s \leq t \leq \lambda_t$,
3. $\lambda'_s = \lambda'_t$.

For example $((3,2,2),1,2)$ is a doubly marked partition, but $((3,2,1),1,2)$ and $((3,2,2),2,1)$ are not.

spt-Crank

To define the spt-crank of a doubly marked partition (λ, s, t), we let

$$g(\lambda, s, t) = \lambda'_s - s + 1,$$

where λ'_s is the number of parts in λ that are not less than s.

spt-Crank

To define the spt-crank of a doubly marked partition (λ, s, t), we let

$$g(\lambda, s, t) = \lambda'_s - s + 1,$$

where λ'_s is the number of parts in λ that are not less than s.

Definition (spt-Crank)

Let (λ, s, t) be a doubly marked partition, and let $g = g(\lambda, s, t)$. The spt-crank of (λ, s, t) is defined by

$$c(\lambda, s, t) = g - \lambda_g + t - s.$$

spt-Crank

The following gives a combinatorial interpretation of $N_S(m, n)$.

spt-Crank

The following gives a combinatorial interpretation of $N_S(m, n)$.

Theorem (Chen, Qi and Zang, [12])

For any integer m and any integer n, $N_S(m, n)$ equals the number of doubly marked partitions of n with spt-crank m.

Partiiton bijections and spt-Crank

We also have the following remarkable theorem.

Partiiton bijections and spt-Crank

We also have the following remarkable theorem.

Theorem (Chen, Qi and Zang, [12])

There is a bijection Δ between the set of marked partitions (μ, k) of n and the set of doubly marked partitions (λ, s, t) of n.

Partiiton bijections and spt-Crank

We also have the following remarkable theorem.

Theorem (Chen, Qi and Zang, [12])

There is a bijection Δ between the set of marked partitions (μ, k) of n and the set of doubly marked partitions (λ, s, t) of n.

Employing the bijection Δ and the spt-crank for doubly marked partitions, one can divide the set of marked partitions of $5n+4$ and $7n+5$ into fiven and seven equinumerous classes.

Andrews-Dyson-Rhoades Conjecture

The following conjecture was posed by Andrews, Dyson and Rhoades [3].

Conjecture

For $m \geq 0$ and $n \geq 0$, we have

$$N_S(m,n) \geq N_S(m+1,n).$$

Andrews-Dyson-Rhoades Conjecture

The following conjecture was posed by Andrews, Dyson and Rhoades [3].

Conjecture

For $m \geq 0$ and $n \geq 0$, we have

$$N_S(m, n) \geq N_S(m+1, n).$$

Andrews, Dyson and Rhoades [3] showed that this conjecture is equivalent to an inequality between the rank and the crank of ordinary partitions.

Andrews-Dyson-Rhoades Conjecture

The following conjecture was posed by Andrews, Dyson and Rhoades [3].

Conjecture

For $m \geq 0$ and $n \geq 0$, we have

$$N_S(m,n) \geq N_S(m+1,n).$$

Andrews, Dyson and Rhoades [3] showed that this conjecture is equivalent to an inequality between the rank and the crank of ordinary partitions.

Dyson and Rhoades [3] found the following connection between the inequality on $N_S(m,n)$ and the inequality on the rank and the crank.

Theorem

Let $N(m,n)$ denote the number of partitions of n with rank m and $M(m,n)$ denote the number of partitions of n with crank m. Then for $m \geq 0$ and $n > 1$, we have

$$N_S(m,n) - N_S(m+1,n) = \frac{1}{2}\left(N_{\leq m}(n) - M_{\leq m}(n)\right). \tag{11}$$

We set

$$M(0,1) = -1, \quad M(-1,1) = M(1,1) = 1, \quad M(m,1) = 0,$$

and define

$$N_{\leq m}(n) = \sum_{|r|\leq m} N(r,n), \tag{12}$$

$\mathrm{M}_{\leq m}(n) = \sum_{|r|\leq m} M(r,n).(13)$

It is clear from (11) that Conjecture 2 is equivalent to the following conjecture.

It is clear from (11) that Conjecture 2 is equivalent to the following conjecture.

Conjecture

For $m \geq 0$ and $n \geq 0$, we have

$$N_{\leq m}(n) \geq M_{\leq m}(n). \tag{14}$$

It is clear from (11) that Conjecture 2 is equivalent to the following conjecture.

Conjecture

For $m \geq 0$ and $n \geq 0$, we have

$$N_{\leq m}(n) \geq M_{\leq m}(n). \tag{14}$$

When $m = 0$, inequality (14) was conjectured by Kaavya [18].

It is easy to check that Conjecture 3 holds for $n = 0$ and $n = 1$. To show Conjecture 3 holds for $n > 1$,

It is easy to check that Conjecture 3 holds for $n = 0$ and $n = 1$. To show Conjecture 3 holds for $n > 1$, Chen, Qi and Zang first gave a reformulation of Conjecture 3 in terms of the rank-set. Then they gave an injective proof of the equivalent inequality.

It is easy to check that Conjecture 3 holds for $n = 0$ and $n = 1$. To show Conjecture 3 holds for $n > 1$, Chen, Qi and Zang first gave a reformulation of Conjecture 3 in terms of the rank-set. Then they gave an injective proof of the equivalent inequality.
Dyson introduced the number $q(m, n)$ of partitions λ of n such that m appears in the rank-set of λ.

It is easy to check that Conjecture 3 holds for $n = 0$ and $n = 1$. To show Conjecture 3 holds for $n > 1$, Chen, Qi and Zang first gave a reformulation of Conjecture 3 in terms of the rank-set. Then they gave an injective proof of the equivalent inequality.
Dyson introduced the number $q(m, n)$ of partitions λ of n such that m appears in the rank-set of λ.
Dyson established the connection between the number $q(m, n)$ and the number of partitions of n with a bounded crank.

To be more specific, let $M(\leq m, n)$ denote the number of partitions of n with crank not greater than m. Dyson obtained the following relation for $n > 1$,

To be more specific, let $M(\leq m, n)$ denote the number of partitions of n with crank not greater than m. Dyson obtained the following relation for $n > 1$,

$$M(\leq m, n) = q(m, n). \tag{15}$$

To be more specific, let $M(\leq m, n)$ denote the number of partitions of n with crank not greater than m. Dyson obtained the following relation for $n > 1$,

$$M(\leq m, n) = q(m, n). \tag{15}$$

Moreover, Dyson proved the following symmetries of $N(m, n)$ and $M(m, n)$:

$$N(m, n) = N(-m, n), \tag{16}$$

$$M(m, n) = M(-m, n). \tag{17}$$

To be more specific, let $M(\leq m, n)$ denote the number of partitions of n with crank not greater than m. Dyson obtained the following relation for $n > 1$,

$$M(\leq m, n) = q(m, n). \tag{15}$$

Moreover, Dyson proved the following symmetries of $N(m, n)$ and $M(m, n)$:

$$N(m, n) = N(-m, n), \tag{16}$$

$$M(m, n) = M(-m, n). \tag{17}$$

Using relations (15), (16) and (17), we found the following connection between $N_{\leq m}(n) - M_{\leq m}(n)$ and $p(-m, n) - q(m, n)$, where $p(-m, n)$ is the number of partitions of n with rank not less than $-m$:

Using relations (15), (16) and (17), we found the following connection between $N_{\leq m}(n) - M_{\leq m}(n)$ and $p(-m, n) - q(m, n)$, where $p(-m, n)$ is the number of partitions of n with rank not less than $-m$:

Theorem

For $m \geq 0$ and $n > 1$, we have

$$N_{\leq m}(n) - M_{\leq m}(n) = 2(p(-m, n) - q(m, n)). \quad (18)$$

It is clear from (18) that Conjecture 3 is equivalent to the following assertion.

It is clear from (18) that Conjecture 3 is equivalent to the following assertion.

Theorem

For $m \geq 0$ and $n \geq 1$, we have

$$q(m,n) \leq p(-m,n). \tag{19}$$

It is clear from (18) that Conjecture 3 is equivalent to the following assertion.

Theorem

For $m \geq 0$ and $n \geq 1$, we have

$$q(m,n) \leq p(-m,n). \tag{19}$$

It is the above theorem that the authors prove in [13].

It is clear from (18) that Conjecture 3 is equivalent to the following assertion.

Theorem

For $m \geq 0$ and $n \geq 1$, we have

$$q(m,n) \leq p(-m,n). \tag{19}$$

It is the above theorem that the authors prove in [13].They prove this by constructing injections from the two sets in the result.

It is clear from (18) that Conjecture 3 is equivalent to the following assertion.

Theorem

For $m \geq 0$ and $n \geq 1$, we have

$$q(m,n) \leq p(-m,n). \tag{19}$$

It is the above theorem that the authors prove in [13].They prove this by constructing injections from the two sets in the result. To do this, they make use of what are called m-Durfee rectangle symbols and some combinatorial analysis.

G. E. Andrews, *The number of smallest parts in the partitions of n*, J. Reine Angew. Math., **624** (2008), 133 – 142.

G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part III*, Springer, New York, 2012.

G. E. Andrews, F. J. Dyson and R. C. Rhoades, *On the Distribution of the spt-Crank*, MDPI Mathematics, **1** (2013), 76 – 88.

G. E. Andrews, F. G. Garvan and J. Liang, *Combinatorial interpretations of congruences for the spt-function*, Ramanujan J., **29** (2012), 321 – 338.

G. E. Andrews and F. G. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc., **18** (1988), 167 – 171.

B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Student Mathematical Library, Vol. 34, Amer. Math. Soc., Providence, RI, 2006.

B. C. Berndt, *What is a q-Series?*, in *Ramanujan Rediscovered: Proceedings of a Conference on Elliptic Functions, Partitions and q-Series in memory of K. Venkatachaliengar: Bangalore*, 1 – 5 June, 2009, N. D. Baruah, B. C. Berndt, S. Cooper, T. Huber, and M. J. Schlosser, eds., RMS Lecture Note Series, No. 14, Ramanujan Mathematical Society, Mysore, 2010, pp. 31 – 51.

B. C. Berndt, H. H. Chan, S. H. Chan, and W. -C. Liaw, *Cranks and dissections in Ramanujan's lost notebook*, J. Comb. Thy., Ser. A **109** (2005), 91 – 120.

B. C. Berndt, H. H. Chan, S. H. Chan, and W. -C. Liaw, *Ramanujan and cranks*, in *Theory and Applications of Special*

Functions. A Volume Dedicated to Mizan Rahman, M. E. H. Ismail and E. Koelink, eds., Springer, New York, 2005, pp. 77 – 98.

B. C. Berndt, H. H. Chan, S. H. Chan, and W. -C. Liaw, *Cranks – really the final problem*, Ramanujan J. **23** (2010), 3 – 15.

H. -C. Chan, *An Invitation to q-Series: From Jacobi's Triple Product Identity to Ramanujan's "Most Beautiful Identity "*, World Scientific, Singapore, 2011.

W. Y. C. Chen, K. Q. Ji and W. J. T. Zang, *The spt-Crank for Ordinary Partitions*, J. Reine Angew. Math., to appear.

W. Y. C. Chen, K. Q. Ji and W. J. T. Zang, *Proof of the Andrews-Dyson-Rhoades conjecture on spt-crank*, preprint (2013).

F. Dyson, *Some Guesses in the Theory of Partitions*, Eureka, **8** (1944), 10 – 15.

R. Fokkink, W. Fokkink and B. Wang, *A relation between partitions and the number of divisors*, Amer. Math. Monthly, **102** (1995), 345 – 347.

F. G. Garvan, *New combinatorial interpretations of Ramanujan's partition congruences modulo* 5, 7 *and* 11, Trans. Amer. Math. Soc., **305** (1988), 45 – 77.

F. G. Garvan, *Combinatorial interpretations of Ramanujan's partition congruences*, in *Ramanujan Revisited*, G. E. Andrews, R. A. Askey, B. C. Berndt, K. G. Ramanathan, and R. A. Rankin, eds., Academic Press, Boston, 1988, pp. 29 – 45.

S. J. Kaavya, *Crank* 0 *partitions and the parity of the partition function*, Int. J. Number Theory, **7** (3) (2011), pp. 793 – 801.

S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.