Construction of the system of real numbers by Cauchy sequences

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In a previous article, we explained the construction of real numbers using Dedekind Cuts; in this article we will explain the construction using Cauchy sequences. A *sequence* of rational numbers is a function $latex {f}&fg=000000$ from the set of positive integers into $latex {\mathbb{Q}.}&fg=000000$ For a positive integer $latex {n}&fg=000000$ the image $latex {f(n)}&fg=000000$ is generally denoted by $latex {f_n,}&fg=000000$ and the sequence itself by $latex {(f_n)_{n=1}^\infty;}&fg=000000$ more informally, we sometimes write the sequence as $latex {f_1, f_2,\ldots;}&fg=000000$ so there is a first term, a second term and so on. Let's take an example. Consider the sequence $latex {(f_n)_{n=1}^\infty}&fg=000000$ of rational numbers, where $latex f_n=(1+1/n)^n$. A few terms of this sequence are (these are approximate values)

$latex \displaystyle
f_1=2.0000,~f_{10}=2.5937,~f_{10^2}=2.7048,~f_{10^3}=2.7169,~f_{10^4}=2.7181.$

It seems like the terms of the sequence are getting closer and closer to something (this something is of course the irrational number $latex {e;}&fg=000000$ but we don't know about it yet). Another important observation is that as we go further down the sequence, the terms are getting closer and closer to each other. This leads to the following idea.

A sequence $latex {(f_n)_{n=1}^\infty}&fg=000000$ of rational numbers is called a Cauchy sequence if for every rational number $latex {\varepsilon>0,}&fg=000000$ there exists a positive integer $latex {N}&fg=000000$ such that $latex {|f_n-f_m|<\varepsilon}&fg=000000$ for all integers $latex {n, m\ge N.}&fg=000000$

We henceforth shall drop "of rational numbers" and simply say "Cauchy sequence". Unlike the Dedekind cuts, distinct Cauchy sequences need not get closer and closer to distinct ``objects". So we need a way of identifying those Cauchy sequences that get close to the same object. Hence we have the following.

Two Cauchy sequences $latex {(f_n)_{n=1}^\infty,~(g_n)_{n=1}^\infty}&fg=000000$ are called equivalent if for every rational number $latex {\varepsilon>0,}&fg=000000$ there exists a positive integer $latex {N}&fg=000000$ such that $latex {|f_n-g_n|<\varepsilon}&fg=000000$ for all integers $latex {n\ge N.}&fg=000000$

To a Cauchy sequence $latex {(f_n)_{n=1}^\infty}&fg=000000$ we associate an object $latex {\mbox{LIM}_{n\rightarrow\infty}f_n,}&fg=000000$ called the *formal limit* of the sequence. For two Cauchy sequences $latex {(f_n)_{n=1}^\infty,~(g_n)_{n=1}^\infty}&fg=000000$ we declare $latex {\mbox{LIM}_{n\rightarrow\infty}f_n=\mbox{LIM}_{n\rightarrow\infty}g_n}&fg=000000$ iff the sequences are equivalent. The set of all such distinct formal limits is denoted by $latex {\mathbb{R}_C.}&fg=000000$

Now we extend the operations $latex {+, \cdot}&fg=000000$ from $latex {\mathbb{Q}}&fg=000000$ to $latex {\mathbb{R}_C.}&fg=000000$ But before that we need the relation $latex {\mathbb{Q}\subset\mathbb{R}_C;}&fg=000000$ like before we need an identification. We identify $latex {p\in\mathbb{Q}}&fg=000000$ with the formal limit of the constant sequence $latex {p, p,\ldots}&fg=000000$ (observe that this is a Cauchy sequence). With this identification we have $latex {\mathbb{Q}\subset\mathbb{R}_C.}&fg=000000$ For real numbers $latex {r=\mbox{LIM}_{n\rightarrow\infty}r_n,~s=\mbox{LIM}_{n\rightarrow\infty}s_n,}&fg=000000$ where $latex {(r_n)_{n=1}^\infty,~(s_n)_{n=1}^\infty}&fg=000000$ are Cauchy sequences, we define $latex r+_Cs :=\mbox{LIM}_{n\rightarrow\infty}(r_n+s_n),~r\cdot_Cs :=\mbox{LIM}_{n\rightarrow\infty}(r_ns_n)&fg=000000$ (i.e., $latex {r+_Cs, r\cdot_Cs}&fg=000000$ are the formal limits of the sequences $latex {(r_n+s_n)_{n=1}^\infty,~(r_ns_n)_{n=1}^\infty}&fg=000000$). We define negation by $latex -r :=(-1)\cdot_Cr.&fg=000000$ Next, we need to define the reciprocal $latex {1/r}&fg=000000$ if $latex {r\ne0.}&fg=000000$ For this, however, we need the following result.

L*et $latex {r}&fg=000000$ be a non-zero real number. Then $latex {r=\mbox{LIM}_{n\rightarrow\infty}t_n}&fg=000000$ for some Cauchy sequence $latex {(t_n)_{n=1}^\infty}&fg=000000$ that satisfies the following property. There exists a rational number $latex {c>0}&fg=000000$ such that $latex {|t_n|\ge c}&fg=000000$ for all $latex {n\ge1.}&fg=000000$*

We then define $latex 1/r :=\mbox{LIM}_{n\rightarrow\infty}1/t_n.&fg=000000$

We now need an order relation $latex {\le_C}&fg=000000$ that will extend the relation $latex {\le}&fg=000000$ on $latex {\mathbb{Q}.}&fg=000000$ A Cauchy sequence $latex {(f_n)_{n=1}^\infty}&fg=000000$ of rational numbers is said to be *positively bounded away from* $latex {0}&fg=000000$ if there exists a rational number $latex {c>0,}&fg=000000$ such that $latex {f_n\ge c}&fg=000000$ for all $latex {n\ge1;}&fg=000000$ it's said to be *negatively bounded away from* $latex {0}&fg=000000$ if $latex {f_n\le-c}&fg=000000$ for all $latex {n\ge1.}&fg=000000$ A real number $latex {r}&fg=000000$ is said to be *positive* (respectively, *negative*) if $latex {r=\mbox{LIM}_{n\rightarrow\infty}r_n,}&fg=000000$ where $latex {(r_n)_{n=1}^\infty}&fg=000000$ is a Cauchy sequence that is positively (respectively, negatively) bounded away from $latex {0.}&fg=000000$ With these definitions it can be proved that every non-zero element of $latex {\mathbb{R}_C}&fg=000000$ is either positive or negative (but not both). For $latex {r, s\in\mathbb{R}_C}&fg=000000$ we can then define $latex {r\le_Cs}&fg=000000$ iff $latex {s-r}&fg=000000$ is positive or $latex {0.}&fg=000000$

With all this $latex {(\mathbb{R}_C, +_C, \cdot_C, \le_C)}&fg=000000$ can be proved to be an ordered field. That it's a complete ordered field takes some effort to prove; it's not as straightforward as the case of Dedekind cuts construction. But one advantage of this construction is that definitions of algebraic operations are straightforward. There is another advantage. Once one defines Cauchy sequences of elements in $latex {\mathbb{R}_C}&fg=000000$ and develops a theory of limits of convergent sequences, the following can be proved. A sequence in $latex {\mathbb{R}_C}&fg=000000$ is convergent iff it's Cauchy. Also, for a Cauchy sequence $latex {(r_n)_{n=1}^\infty}&fg=000000$ of rational numbers, $latex {\displaystyle\lim_{n\rightarrow\infty}r_n=\mbox{LIM}_{n\rightarrow\infty}r_n,}&fg=000000$ i.e., formal limits are actual limits. This system $latex {(\mathbb{R}_C, +_C, \cdot_C, \le_C)}&fg=000000$ also has $latex {(\mathbb{Q}, +, \cdot, \le)}&fg=000000$ as a subsystem.

We have apparently obtained two distinct complete ordered fields: $latex {(\mathbb{R}_D, +_D, \cdot_D, \le_D)}&fg=000000$ and $latex {(\mathbb{R}_C, +_C, \cdot_C, \le_C),}&fg=000000$ each containing the system $latex {(\mathbb{Q}, +, \cdot, \le)}&fg=000000$ of rational numbers. However, it can be proved that any two complete ordered fields are isomorphic. So essentially, the system of real numbers is unique in this sense. Apart from these there are many other approaches. Real numbers can be developed via slopes or almost homomorphisms, or via continued fractions, or via alternating series, or from hyperrational numbers, or from surreal numbers etc. It's like looking at an object in visible light, or ultraviolet, or infrared, or in some other electromagnetic wave of a different frequency; each frequency showing a different aspect of the object. The object however remains the same - the system of real numbers.
