Preface

A grand challenge is to create an environment which makes enjoyable self learning a normal part of education. This book is written to support such an enterprise for school mathematics. We have practical suggestions on how this can be done. The topics we cover are listed in the table of contents. The special feature of the book is that we start from counting numbers and then gently guide the reader into the world of mathematics. We hope teachers, students and all who have an interest in understanding mathematics will find our book useful and perhaps even entertaining.

Enjoyable self learning is possible. We see that this is exactly what very young children love to do. They enjoy themselves. They feel, touch, taste and experiment with things around them. They are eager to learn. They want to join in and be part of the exciting world they belong to. The normal environment for most young children is very supportive and encouraging for learning. Mistakes are tolerated and progress made is praised. The lesson we learn from this is that to encourage learning a supportive, encouraging environment of the kind we had when we were young, would help. Such an environment would help in many ways. It would make a person eager to learn, willing to make the effort required to progress and feel confident that difficulties on the way would be overcome.

Can such an atmosphere of self learning be created for all stages of life? Maria Montessori showed how such an atmosphere could be created for the primary school years. She demonstrated by her work that children want to learn and that they also have a love of order. By using these traits present in children Montessori went on to create an environment and support system that helped children get on with the enjoyable task of learning on their own. She developed this environment and support system based on her close observations, reflections and experimentations on the way children learn. The role of the teacher in this approach was to provide every child with the background of skills necessary to enable them to confidently travel along the road of self learning. The teacher was a facilitator.

The current ongoing information and technological revolutions are changing the world at a rate which is unprecedented. In this constantly changing world the relevance of self learning for education is becoming very important. As great changes happen new ways of doing old tasks are found. We see tha the way people buy and sell goods or even socialise is changing fast. It is becoming global and dependent on the existence of fast communication, se-

cure ways of transferring funds and fast ways of getting information. These changes are challenging and stressful but at the same time they are opening up new opportunities. To cope with these changes young people must learn to adjust to new situations quickly and be used to learning things on their own. Increasingly they would need to master new idea and new technologies and acquire new skills on a ongoing basis. Learning is thus becoming a lifelong activity. In order to be able to do this effectively students should have got used to learning on their own while in school. Learning for enjoyment and for work will then be natural. Those without this skill will be at a disadvantage.

The way this ability is to be taught for all the stages of life is not clear. The environment for self learning we realise will be different at different times for different age groups and for different cultures. But some basic features are very clear. We have already listed some of the features that a learning environment should have. This list of requirements must also take into account an important fact: Human beings naturally learn and adapt to change as they live. They also effortlessly learn and remember things that interest them. The urge to learn and pick up new ideas and use them is present in all human beings. Every moment we experience new situations to which we respond and from which we learn. This natural in built skill for learning present in human beings needs to be understood and used in education.

Certain kinds of learning, however, such as learning to read, learning music , learning a craft or learning mathematics where new ideas, new methods and new concepts appear, need a more structured approach. These are subjects and skills that are often difficult to learn without assistance. Often the new concepts to be mastered are abstract. They do not seem to be present in our day to day environment. Such concepts cannot be acquired in a ad hoc way. They need guides or mentors to explain the reason for learning them and to explain the advantages gained by mastering them. The advantages from learning can vary from economic gain, greater opportunities for advancement in a profession to acquiring the ability to help others or to understand the mysteries of the world or to appreciate and create music or dance or poetry or art. A good mentor can make a student realise that a vast range of possibilities are out there and perhaps give them a glimpse of the world of ideas or the world of beauty that is there for them to explore. The guide can also point out pitfalls and difficulties on the way so that students can progress in their learning with confidence. Each student have the

experience of coming in contact such a mentor.

But even with a good mentor or teacher to instruct students the underlying aim of education should remains to encourage students to learn on their own with joy. Because at the end it is the student who has to make the effort to master a concept or acquire a skill. Understanding concepts and mastering skills ultimately are personal transformational experiences for each student. If such an experience is joyful it will encourage the student to continue with confidence. An environment where learning is joyful will thus allow the talent of students to flourish, will enrich the life of students and open doors of opportunity for them. Students should know that in order to learn mistakes have to be made. Einstein once said that the only time you do not make mistakes is when you are dead.

A good learning environment should also make use of all the senses of the student. Ideally instruction given should make creative use of art, drama, dance, stories and weave them all to illuminate what is being discussed. Such a multi layered approach should be done for all subjects, including mathematics.

Can this be done? We do not know. We have listed some general requirements but the challenge remains how to turn those aspirations into a practical set up that can be implemented. The way forward is not clear. It is a challenge. But there are suggestions from wise people from the past as well as hints from modern day educational research which are helpful.

The practical environment needed to encourage joyful self learning with an urge to understand and to solve problems for later stages of life has been created in small exclusive groups, in different countries at different times. The great Universities of the world are examples of places where such an environment exists. These places have a tradition of creativity. They are filled with people who have a passion for their specialty who want to know and understand and explain. They imaginatively transfer knowledge from one generation to the next. This environment attracts young people who flock to these great centres eager to learn, to reflect and to create. These young people are willing to work hard and are willing to dedicate themselves trying to understand the mysteries of nature or to understand the way human beings live and interact or to create works of music or art or dance that give joy to all and enrich our lives. By their efforts these talented young people fulfill their dreams , learning from and interacting with creative people.

The key requirement for making progress in self learning is thus to have an environment that is joyful, which encourages creativity and makes all

students eager to learn.

Where there is creation there is progress, where there is no creation there is no progress. Where there is joy there is creation, where there is no joy there is no creation.

Is it possible to construct such an atmosphere without the presence of great creators and innovators? The answer is probably not. But with modern day technology and instant communications it might be possible to build a virtual global learning centre where young students get the opportunity to listen to and interact with creative knowledge makers no matter where they live. By these means the aspiration of these young people will become global. They will see themselves as members of a global community. Their standards will then be guided by global standards. Their aspirations will be global aspirations. Joseph Campbell, the great student of myths and legends, reflecting on the rise and fall of civilisations said: "The rise and fall of civilisations in the long, broad course of history can be seen to have been largely a function of the integrity and cogency of their supporting cannons of myths for not authority but aspiration is the motivator , builder and transformer of civilisation.." Thus if most members of a society have a positive attitude towards learning which leads them to having ambitious aspirations then that society will represent a civilisation on the rise. Aspirations are important.

We want to implement these ideas for mathematics learning for students of senior school. We want to make a start on this journey of promoting self learning. The present book is a first step.

Our aim is to help students to be not just successful in life by increasing their quantitative and mathematical skills but also to make them experience the joy of understanding great creative ideas and to reflect on and experiment with the key concepts of mathematics. Such an understanding will not only help the student to tackle mathematics examinations with confidence but will make them aware of the potential uses of the subject creatively in everyday life as well. They would have acquired a life skill.

How do we aim to do this? We feel that one essential step is to present mathematics as an enjoyable human activity created by people with great imagination who have a passion to solve problems and find patterns. We feel that everyone is capable of learning and enjoying mathematics. Perhaps different students with different interests will make progress at different rates. To create interest we feel that the beauty of mathematical ideas as well their usefulness in everyday life should be pointed out through examples. Interest can come from hearing stories about some of the creators of the

subject, explaining how they solved problems, and by by giving examples of challenging unsolved problems. Interest can also come from using examples from areas that excite the imagination of the student.

We hope to progress to using music, dance, art, and drama to illuminate key mathematical concepts. In order to this well we will need ideas and suggestions from all interested in mathematical education, specially from students.

We also feel it is important that students should appreciate the power of mathematics as a great enabling tool. Such an understanding can come from using mathematics creatively to solve problems of interest to students.

For some students the austere beauty of mathematics beckons them to explore the subject, for some the practical utility of mathematics to solve everyday problems makes the subject exciting while for some others the patterns of mathematics when they show up in other disciplines is a joyful experience. Sadly, however, for most students mathematics is an imposition. It is something they have to do and that it is of no use in the real world and it is of no interest to them. Part of the reason for this attitude is the way mathematics is taught in school as a collection of rules for solving certain examination problems. This way of teaching is equivalent to teaching English without poetry or stories or plays but simply through the rules of grammar and spelling drills.

The richness of mathematics can only be fully revealed to students by a gifted teacher who gets the student to experience this richness and thus makes the subject come alive. When this happens the student can get further inspiration from reading books.

Mathematics has three faces. It has a creative face This face reflects clever imaginative ideas due to people from all over the world. The subject is a global adventure of exploration and discovery. The creators of mathematics were and are "dreamer of dreams" who discover patterns and came up with ways of looking at the world which were enlightening, novel and useful. This process of discovering and studying significant patterns continues. It is indeed accelerating. Here the motivation is to study the mathematical world guided by ideas of truth, beauty and elegance.

The great Russian mathematician Gelfand said mathematics has four traits in common with music, and other creative arts and sciences, the first is beauty, the second is simplicity, the third is precision, and the fourth ...crazy ideas.

Then there is the practical face of mathematics. As knowledge advances

the citizens of a country increasingly need to understand ways of reasoning with numbers in order to digest the bewildering amount of information that they receive everyday. Thus the practical face of mathematics does not require the citizen to have mastery over specific techniques but rather to have the confidence to deal with numbers, to be able absorb new ideas and assess the merit of arguments and information presented to them. The boundary of what a citizen needs to know is constantly shifting. The primary training for the citizen is thus one that gives them with a positive attitude towards continuing learning, gives them a strong belief in the power of rational reasoning, and gives them mastery of a set of basic quantitative skills. They should also have a strong sense of self belief. A "Yes I can" mentality. The educational system of a country should ideally equip its citizens with these life skills.

Then there is the enabling face of mathematics.This face has to do with the imaginative and creative use of mathematics to solve problems in different areas of science, computer science, engineering, finance, economics, medicine,and elsewhere. Without mathematics progress in these disciplines would suffer. The Philosopher Whitehead reflected on the paradox that to solve the most practical of problems very often required the most abstract of ideas. The use of quaternions in the functioning of the GPS system used for navigation is such an example.

The students of mathematics should thus be able to use mathematics confidently for his or her practical day to day needs but should also get a taste of the joy of solving worthwhile problems, experience the pleasure of understanding important ideas during their education and read about some of the creators of the subject. Young people educated in this way will be a valuable pool of talent. These young people will be self confident and will think it natural to use mathematics creatively to tackle major problems of the world. Learning with confidence, joyfully, on their own will have opened new horizons for these students and will make them see the world differently.

But the first step the student has to take in order to start on this enjoyable journey of discovery is to learn to work confidently with numbers and symbols. The only way to get this confidence is by practice. To become confident with numbers it might be helpful to learn to work out simple multiplication or division problems by breaking up numbers into convenient bits. For instance to work out 6×9 we might replace the 9 in the problem by $10 - 1$. Then the multiplication problem becomes easier. We have

$$6 \times 9 = 6 \times (10 - 1) = 60 - 6 = 54$$

Working out more complicated problems without using a calculator by breaking up numbers in clever ways will make students get to know numbers and to be more at ease when working with them.

Working confidently with symbols is the next hurdle that has to be overcome. This can happen if the power of using symbols for solving problems becomes clear to the student. They should understand well that the great power of mathematics lies in its ability to present information in a condensed way, with precision, using symbols. It is a language.

The power of even the well known symbols such as $+, -, \times, \div$ for adding, subtracting, multiplying and dividing should be explained. These symbols were invented over many centuries. The ones we now use were firmly established only as late as the 17th century.

The other great innovation in mathematics was the introduction of a symbol x to represent something we do not know. This is an idea of great power and needs to be presented to the student as a "giant step for mankind."

Confidence with numbers and understanding of the power and usefulness of symbols are essential first steps which then allow a student to start on a journey of exploring and perhaps discovering some of the treasures of mathematics.

This initial step of getting to know numbers and symbols is equivalent to building a vocabulary of ones own in a new language so that one can read interesting books or to write creatively. Help to motivate oneself for this stage of learning can come from reading and hearing the great stories of the creators of the world of mathematics and from case studies of mathematical achievements. But practice and effort are needed as well.

Current research in mathematical education has identified certain essential features that help students learn and enjoy the subject.

1. It has been established that students learn best when they understand why something is taught and why it is important.

2. It has been established that understanding improves when students are able to discuss their ideas freely with other students.

3. It has been established that understanding improves when students work in groups. But some in a group might like to be left alone to think and dream. This fact needs to be remembered.

4. It has been established that students progress best when they are asked to work on challenging problems of interest to them.To solve problems effectively students need time to think and reflect.

5. It has been established that while drill and practice to master a technique are essential asking a weaker student to keep redoing the same kind of problem does not help. The underlying difficulty and the reason for the lack of confidence and interest need to be identified. Once this is done there should be rapid improvement in learning.

In groups the students can learn, reflect and teach each other. This Trinity of learning, reflecting and teaching is a an essential part of education.

No one can be taught to understand a concept. Everyone learns and understands concepts on their own. Understanding happens suddenly. Suddenly a student thinking about a problem stops and says "Got it" . The process by which this happens is not known. It is a creative act which can only happens if the student feels that the concept is really important so that he or she is willing to make the effort of concentration required to understand the new idea presented to them. This effort of concentration becomes natural if the environment of learning is encouraging, and the student feels that he or she can master the concept and he or she is interested enough to make the necessary effort.

There are other features of understanding that need to be remembered. The great French mathematician Poincare emphasised the essential role that elegance and beauty play in the creation of mathematics. Very often students never get see these aspect of the subject in their school mathematical education. It is important that they do.

Summarising: our aim is to contribute to a great enterprise of education namely to make learning a joyful experience. For mathematics we feel this can happen by making students realise that mathematics is a living subject which is enjoyable, that mathematics can be used to help humanity and that it opens many doors of opportunity for them. We hope that with such a positive attitude towards learning students will use mathematics and anything else that they learn creatively in whatever they choose to do later in life.

Our big aim is thus to prepare students for the adventure of life by making them realise that at the end it is they who have to make the effort to learn. Our immediate aim is to help students, teachers and others who want to learn or teach basic mathematics by providing a text in which simple things are explained, where there is some history, where the reason why certain operations of mathematics work is explained by pictures and words, where some great ideas of the subject are presented and where some of the great unsolved challenge problems of the subject are listed. By these means we hope that learning mathematics will become enjoy

Siddhartha Sen
Dublin, May 2012

Acknowledgment