An Invited Talk On

Fundamental Theorem of Algebra

delivered by

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Fundamental Theorem of Algebra

§1   What the theorem is?

§2   History of Fundamental Theorem of Algebra.

§3   Proofs of FTA
- **Complex Analysis**
- **Topology**
- **Algebra**

§4   A Visual Approach

§1 What the theorem is?

- **<u>Statement:</u>** Every non-constant polynomial with complex coefficient has a root in the set of complex numbers.

  <u>Equivalently</u>

  Every polynomial of degree $n >= 1$ over the field of complex numbers has n and only n roots.

  <u>Equivalently</u>

  Every polynomial of degree $n >= 1$ over the field of complex numbers can be expressed as

  $$a(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)$$

  Thus if

  $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ is a polynomial, where the coefficients $a_0, a_1, \ldots, a_{n-1}$ or real or complex numbers, then $\ni$ a complex number α such that $p(\alpha) = 0$.

Contd…

Equivalent formulations of the problem are the following:

I   Every non-constant polynomial with real coefficient can be expressed as product of real linear or real quadratic factors i.e. every polynomial with real coefficients has a complex root.

II For each n>=1, every n×n square matrix over $\mathbb{C}$ has an Eigenvector (hence an Eigen value) i.e. every linear operator on an n-dimensional complex vector space has an Eigenvector.

Contd…

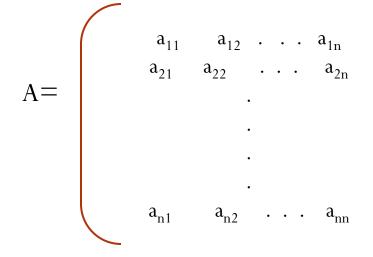- **<u>The arguments are the following</u>**

FTA↔I

Because if FTA is proved, then any polynomial with real coefficients can be treated as a polynomial with complex coefficients and can be therefore factored into quadratic & linear factors and also realizing that any real quadratic will have complex roots in conjugate pairs.

<u>Again</u> if I is proved, then given any polynomial p(z) with complex coefficients, then one can construct a polynomial with real coefficients as $p(z)\overline{p(z)}=0$ i.e.

$|p(z)|^2=0$. Now any root of the resulting polynomial will either be root of the original polynomial or the complex conjugate of a root.

Now FTA↔II

Suppose FTA is proved. Now suppose

Contd…

$$A=\begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & & . & . & . & a_{2n} \\ & & & . & \\ & & & . & \\ & & & . & \\ & & & . & \\ a_{n1} & a_{n2} & & . & . & . & a_{nn} \end{pmatrix}$$ is any n×n square matrix over $\mathbb{C}$

Consider

$$|zI-A|=0$$

i.e.

$$\begin{vmatrix} z-a_{11} & -a_{12} & . & . & . & -a_{1n} \\ -a_{21} & z-a_{22} & . & . & . & -a_{2n} \\ & & . & \\ & & . & \\ & & . & \\ -a_{n1} & -a_{n2} & . & . & . & z-a_{nn} \end{vmatrix} = 0$$

Contd...

Which is

$$f(z)=z^n+a_{n-1}z^{n-1}+\ \dots\ +a_1z+a_0=0$$

If λ is a root of the equation then $\lambda I\text{-}A$ is singular. Therefore

$$\exists x\neq0:\ (\lambda I\text{-}A)x=0 \qquad i.e.\ Ax=\lambda x$$

<u>Conversely</u> suppose II is true. Then we consider any polynomial

$$f(z)=z^n+a_{n-1}z^{n-1}+\ \dots\ +a_1z+a_{0,}\ n>=1$$

& $a_j \in ¢$

<u>Claim by induction</u>

$$\det[\lambda I_n\text{-}A]=f(\lambda)\ \text{where}$$

Contd...

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & -a_0 \\ 1 & 0 & 0 & 0 & \ldots & -a_1 \\ 0 & 1 & 0 & 0 & \ldots & -a_2 \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & \cdot & & \\ 0 & 0 & 0 & 0 & \ldots 1 & -a_{n-1} \end{pmatrix}$$

For n=1 consider $f(z) = z + a_0$

& take $A = [-a_0]$

Contd....

$|\lambda I - A| = |\lambda I + a_0| = \lambda + a_0 = f(\lambda)$

<u>For n=2</u>

$$f(z) = z^2 + a_1 z + a_0$$

$$A = \begin{pmatrix} 0 & -a_0 \\ 1 & -a_1 \end{pmatrix}$$

$$|\lambda I_2 - A| = \begin{pmatrix} \lambda & a_0 \\ -1 & \lambda + a_1 \end{pmatrix} = \lambda^2 + a_1 \lambda + a_0 = f(\lambda)$$

Suppose for n-1 the result has been proved. Then if

$$f(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

$$f(\lambda) = \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0$$

Contd….

- Now if

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{pmatrix}$$

Then

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & \ldots & 0 & a_0 \\ -1 & \lambda & \ldots & 0 & +a_1 \\ & & \cdot & & \\ & & \cdot & & \\ 0 & 0 & \ldots & -1 & \lambda + a_n \end{vmatrix}$$

Contd…

$$= \lambda \begin{vmatrix} \lambda & 0 & \dots & 0 & a_1 \\ -1 & \lambda & \dots & 0 & a_2 \\ & & \cdot & & \\ 0 & 0 & \dots & -1 & \lambda+a_{n-1} \end{vmatrix} +$$

$$(-1)^{n-1}a_0 \begin{vmatrix} -1 & \lambda & \dots & 0 \\ 0 & -1 & \lambda & \dots & 0 \\ & & \cdot & & \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix}$$

Contd...

$$= \quad \lambda \begin{vmatrix} \lambda & 0 & \ldots & 0 & a_1 \\ -1 & \lambda & \ldots 0 & a_2 & \quad + a_0 \\ & & & . \\ 0 & 0 & \ldots & -1 & \lambda+a_{n-1} \end{vmatrix}$$

$= \lambda [\lambda^{n-1}+a_{n-1}\lambda^{n-2}+ \ldots +a_1]+a_0$

$= \lambda^n+a_{n-1}\lambda^{n-1}+ \ldots +a_{1\lambda}]+a_0$

$= f(\lambda)$

Therefore

$\quad det[\lambda I-A]=f(\lambda)$

Therefore if A has a characteristic vector then for some λ $det[\lambda I-A]=0$

Therefore $f(z)=0$ has a root in $\not\subset$.

History of Fundamental Theorem of Algebra

Some versions of the statement of Fundamental Theorem of Algebra first appeared early in the 17th century in the writings of several mathematicians including **Peter Roth, Albert Girard** and **Rene Descartes.**

All these mathematicians believed that a polynomial equation of degree n must have n roots & the only problem was, they believed to show that these roots lie in the set of complex numbers.

- **1702:** A proof that the FTA was false was given by **Leibnitz** in 1702 when he asserted that $x^4 + t^4$ could never be written as a product of two real quadratic factors. His mistake came in not realizing that $\sqrt{i}$ could be written in the form $a+ib$, where a and b are real.

- **1742:** Euler in the year 1742 in correspondence with **Nicolaus II, Bernouli** and **Goldbach** showed that the **Leibnitz** counter example was false.

- **1746: Jean Le Rond d' Alembert** in the year 1746 made the first serious attempt at a proof of the FTA. He published his proof in "Recherches sur la Calcul integral, Hist. Acad. Sci. Berlin 2(1746),182-224".

    Proof given by d'Alembert was not very rigorous and had several weaknesses. He used the following Lemma:

D'Alembert's Lemma

Suppose f is a non constant polynomial and $f(z_0)\neq0$. Then for every $\epsilon>0$ there is some z such that

$$|z\text{-}z_0|<\epsilon \quad \text{and} \quad |f(z)|<|f(z_0)|$$

D'Alembert's proof of this lemma was not very rigorous and it was unnecessarily complicated (A simpler proof of the lemma was given by **Jean Robert Argand** in 1806).

Furthermore he also used that a continuous real-valued function on a compact set achieves a min. value, a fact that had yet not been rigorously proved in D'Alembert's lemma. Therefore proof given by d'Alembert in 1746 was not convincing.

- **1749**: Shortly after d'Alembert's proof, **leonhard Euler** published an algebraic proof of the FTA by attempting to prove that '*Every polynomial of the nth degree with real coefficients has precisely n zeros in ₡*'.

- **1772: Joseph-Louis Lagrange** in the year 1772 raised many objections to Euler's proof and pointed out many gaps. He used his knowledge of permutation to fill all the gaps in Euler's proof.

  However one significant gap still remained. Euler and Lagrange both assumed that a polynomial of degree n would have n roots and that the only thing to be proved is that these roots are complex numbers.

- **1795: Laplace** in the year 1795 tried to prove the FTA using a complete different approach using the discriminant of a polynomial. His proof was quite elegant but <u>Unfortunately</u> in his proof also, the existence of roots was assumed.

1799: The first person to notice these gaps was Carl Friedrich Gauss. He is credited with producing the first correct proof in his doctoral dissertation, which he submitted in the year 1799.About Euler-Lagrange proof Gauss says the following:

'If one carries out operations with impossible roots, as though they really existed and says for example the sum of all the roots of the equation

$$x^m + ax^{m-1} + bx^{m-2} + \ldots\ldots = 0$$

is equal to –a even though some of them may be impossible to exist (which really means: even if some are non existent and therefore missing) then I can only say that I thoroughly disapprove of this type of Argument'.

Gauss himself did not claim to give the first proper proof. He merely called his proof new.

About D'Almbert's proof, despite his objections he said:

'A rigorous proof could be constructed on the basis of D'Alembert's lines'.

Gauss proof of 1799 was topological in nature. However, it does not meet our present day standards of a rigorous proof.

- **<u>1814</u>**: In 1814 **jean Robert Argand** published a proof based on D'Alembert's 1746 idea and was the simplest of all the proofs. Gauss himself throughout his life time kept on working over FTA and two years after Argand's proof, Gauss in 1816,published a second proof which was complete and correct. In 1816 itself, Gauss gave third proof of the FTA and in 1849,on the 50th Anniversary of his first proof gave the fourth proof of the FTA. Since, then (& even before) many mathematicians across the globe have been working over the theorem and have been publishing various proofs using tools from various branches of mathematics, More than **100** proofs have come up so far and the recent most in **2012** itself is entitled.

- **<u>2012:</u>** "Some Riemannian geometric proofs of the Fundamental Theorem of Algebra", *Differential Geometry - Dynamical Systems* Almira, J.M.; Romero, A. (2012), **14**: 1–4

Many mathematician who got attracted by the problem and contributed to the solution of the FTA, include the following:

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Proofs of Fundamental Theorem of Algebra

All proofs of the FTA necessarily involve some analysis or more precisely the concept of continuity of real or complex polynomials.

We mention, broadly three different approaches

➢Complex Analysis

➢Topology

➢Algebra

1st Proof (via Complex Analysis) (R P Boas Amer. Math. Monthly 1964])

- **This proof is based on the use of a classical theorem of Picard.**
  - Picard's Theorem: if there are two distinct values in the complex plane, which a given entire function never assumes, then the function is constant.

  I Proof of fundamental theorem of algebra

  Suppose $f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$, $n >= 1$ & a_i's are constants.

  Let if possible $f(z) \neq 0$ for any z in $\mathbb{C}$.

Contd...

<u>Claim</u>   f(z) fails to take on one of the values

$$1/k, \qquad k=1, 2, \dots$$

Suppose for each k

$$f(z_k)=1/k \qquad k=1,2,3, \dots$$

Then, we have

$$|f(z)| = |z|^n \left| 1+a_{n-1}/z + a_{n-2}/z^2 + \dots + a_0/z^n \right|$$

$\therefore$ $|f(z)| \dashrightarrow \infty$ as $|z| \dashrightarrow \infty$

$\therefore$ $\exists R>0 \colon |f(z)|>1 \quad \forall z \colon |z|>R.$

Contd...

But $\{z: |z| \leq R\}$ is closed & bounded. $\therefore$ these (z_k) have a limit pt. i.e. $\exists$ a subsequence (zn_i):

$$zn_i \text{ ----> } z \in \{z: |z| \leq R\}.$$

As f is continuous

$$f(z) = lt\ f(zn_i) = lt\ (1/kn_i)=0$$

$\therefore$ **f(z) fails to take on the value 1/k for some k.**

Now $f(z) \neq 0$ for each z

and

$f(z) \neq 1/k$ for each z

and f is entire. $\therefore$ f must be a constant by Picard's theorem, a contradiction because degree of $f(z) >= 1$.

Hence $\exists\ z_0 \in \mathbb{C}$ such that $f(z_0)=0$.

IInd Proof (Topological Proof)

Suppose that the polynomial

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z^1 + a_0 , a_n = 1$$

has no root, so that for every complex number z

$$f(z) \neq 0.$$

On this assumption, if we now allow z to describe any closed curve in the x, y - plane, f(z) will describe a closed curve Γ which never passes through the origin.

Definition: We define the *order* of the origin *O* with respect to the function f(z) for any closed curve C as *the net number of complete revolutions made by an arrow joining O to a point on the curve Γ traced out by the point representing f(z) as z traces out the curve C.*

Contd…

<u>Pf continued</u> For the curve C we take a circle with O as center and with radius t. We define the function $\varphi(t)$ to be the order of O with respect to the function $f(z)$ for the circle about O with radius t. Then $\varphi(0) = 0$, since a circle with radius 0 is a single point, and the curve Γ reduces to the point $f(0) \neq 0$.

<u>Claim</u> $\varphi(t) = n$ for large values of t.

Consider a circle of radius t such that

$$t > 1 \text{ and } t > \{ |a_0| + |a_1| + \ldots + |a_{n-1}| \}$$

we have if z lies on this circle, then $|z|=t$ and then

Contd...

$$|f(z) - z^n| = |a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_0|$$
$$\leq |a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-2} + ... + |a_0|$$
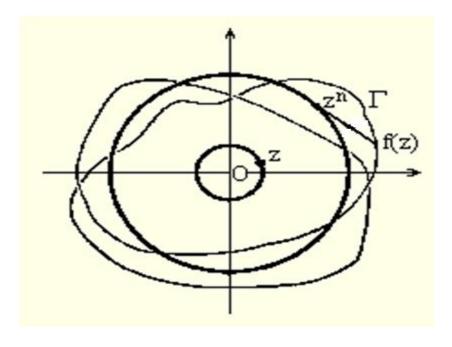$$= t^{n-1}[|a_{n-1}| + |a_{n-2}| / t + ... + |a_0| / t^{n-1}]$$
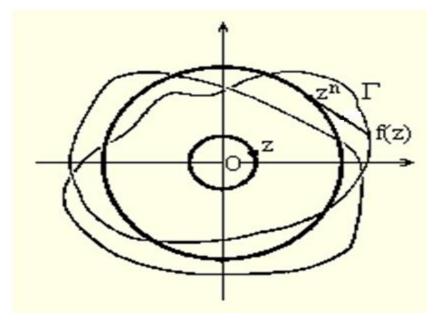$$\leq t^{n-1}[|a_{n-1}| + ... + |a_0|]$$
$$< t^n = |z|^n.$$

i.e.

$$|f(z) - z^n| < |z|^n = |z^n|.$$

Contd...

Since the expression on the left is the distance between the two points z^n and $f(z)$, while the expression on the right is the distance of the point z^n from the origin, we also see that the straight line joining the two points $f(z)$ and z^n cannot pass through the origin so long as z is on the circle of radius t about the origin.

Contd...

This being so, we may continuously deform the curve traced out by *f(z)* into the curve traced out by z^n without ever passing through the origin, simply by pushing each point of *f(z)* along the line segment joining it to z^n. But the order of the origin will vary continuously and can assume only integral values during this deformation, it must be the same for both the curves. Since the order for z^n is *n*, the order for *f(z)* must also be *n*.

Contd...

But the order $\varphi(t)$ depends continuously on t, since f(z) is a continuous function of z. Hence we shall have a contradiction, for the function $\varphi(t)$ can assume only integral values & therefore cannot pass continuously from the value 0 to n.

[This proof was published in the book:

What is Mathematics by R. Courant and H.Robbins, published by Oxford University Press in 1966. The proof originally appears in American Math. Monthly 42 (1935), p.501-502].

III Proof (Algebra)

Proof based on linear algebra rests on the following

Lemma1:- For each odd n≥1, every n*n matrix over R has a real eigenvector.

Equivalently for each odd n≥1, every linear operator on an n-dimensional real vector space has an eigenvector.

Pf:- Any odd degree polynomial over R has real roots by Intermediate value theorem.

Lemma2:- For each odd n ≥ 1, any pair of commuting linear operators on an n-dimensional real vector space has a common eigenvector.

Pf:- We prove the result by odd dimension n.

<u>For n=1</u>

The result is trivially true because $A_1=\alpha I$ and $A_2=\beta I$

Therefore any non-zero vector in a one-dimensional space is a common eigenvector of A_1 & A_2.

Contd...

Suppose n>1 & suppose we have settled the case for odd dimension less than n. Let A_1 & A_2 be two commuting linear operators on a real vector space V of dimension n. By Lemma 1 , A_1 has a real eigenvalue say λ. Let

U & W be subspaces of V invariant under A_1 dim W≥1 since λ is an eigenvalue of A_1. As A_1& A_2 commute U & W are invariant under A_2 also.

Therefore one of U & W has odd dimension. Two cases arise:

Case–I:– If the subspace with odd dimension is U. Then as dim W $\geq$ 1 therefore dim U<n &

$\qquad$ A_1: U--->V &

$\qquad$ A_2: V--->U

have common eigenvector in U & therefore have a common eigenvector in V and we are done.

Contd...

Case-II:- if the subspace with odd dim is W. Then if dim W $=$ dim V

Then W$=$V & therefore $A_1=\lambda I$ and

therefore any eigenvector of A_2 in V would also be an eigenvector of A_1.

If dim W $<$ dim V$=$n then by assumption A_1 & A_2 have common eigenvector in W and therefore common eigenvector in V and we are done again.

Remark:- Lemma 2 does not say that A_1 & A_2 have a common eigenvalue, but rather a common eigenvector. A common eigenvector does not have to occur with the same eigenvalue. e.g.

let

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Contd...

on $R^3 = V$. Then $A_1 A_2 = A_2 A_1$ and therefore $A_1 A_2$ must have a common eigenvector. One common eigenvector is

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

with eigenvalue 1 for A_1 & 3 for A_3. In fact, this is the only common eigenvector for A_1 and A_2 in R^3 (up to scalar multiple).

The next lemma generalizes <u>Lemma1 & Lemma2.</u>

Contd...

<u>Lemma 3</u> Fix a power of 2 say 2^k and a field F. Suppose that for every vector space V over the field F, whose dimension is not divisible by 2^k, every linear operator on V has an eigenvector. Then for every vector space V over F whose dimension is not divisible by 2^k, any pair of commuting linear operators on V has a common eigenvector.

([Note that if F=R and $2^k=2$, it becomes Lemma2.])

<u>Proof:-</u> we prove the result by induction on the dimension of the vector space. If dim V=1, then we are trivially done.

Suppose dim V=d>1 is not divisible by 2^k. Suppose, we have settled the case for dimension less that d, which are not divisible by 2^k. Now let A_1 & A_2 be commuting linear operators on V, where dim V=d. By assumption A_1 has an eigenvector say λ in F.

Contd...

Let

$$U = \operatorname{Im}(A_1 - \lambda I), \quad W = \ker(A_1 - \lambda I).$$

These are subspaces of V and dim W≥1. Also

$$A_1(U) \subset U \text{ and } A_1(W) \subset W.$$

Since A_1 & A_2 commute $A_2(U) \subset U$

$$A_2(W) \subset W$$

Now dim U + dim W =d which is not divisible by 2^k

∴ at least one of U & W has dimension which is not divisible by 2^k.

Two cases arises:

Case 1:- if dim V is not divisible by 2^k. Since dim W>=1 ∴ dim U< d. Therefore by induction hypothesis A_1 & A_2 have a common eigenvector in U & ∴ in V and we are done.

Case 2:- If dim W is not divisible by 2^k Now if dim W=d, then W=V & $A_1 = \lambda I$. Therefore any eigenvetor of A_2 is also an eigenvector of A_1 and

If dim W< d, then by assumption A_1 & A_2 have a common eigenvector in W. Therefore in V and we are done again.

Contd...

Now proof of fundamental theorem of algebra.

Let A be an n*n matrix over C . Write n = $2^k n^1$, where k>=0 & n^1 odd.

Suppose k=0 ∴ n = n^1 is odd.

Let   H = { T∈ M_n (C) : T^*=T }

Where T^* denotes the conjugate transpose of T & M_n (C) denotes the set of all n*n matrices over $\mathbb{C}$.

(dim M_n (C) =n^2 over C)

H is a real vector space & dim H=n^2 over R. As n is odd n^2 is also odd. Thus dim H is also odd.

Let B be any element of H. Let

T=AB Then T=AB=(AB + (AB)*)/2 + i (AB-(AB)*)/2i

$\qquad\qquad$ =(AB+BA*)/2 + i (AB-BA*)/2i

Contd...

Now $L_1 : H \longrightarrow H$ be defined by

$L_1(B) = (AB + BA*)/2$ &

$L_2(B) = (AB-BA*)/2i \quad \forall B \in H$

We can see that $L_1(B) = [(AB + BA*)/2]*$

$= (AB+BA*)/2$ and also

$L_2(B) = [(AB-BA*)/2i]*$

$= (AB-BA*)/2i$ and also L_1 and L_2 commute.

$\therefore$ as dim H= odd & H is a real vector space therefore L_1 & L_2 have a common eigenvector in H say B i.e. $\exists$ B$\neq$0 in H such that

$L_1(B) = \lambda_1 B$

$L_2(B) = \lambda_2 B$

for some λ_1 & λ_2 real.

Contd...

Hence

$AB = L_1(B) + iL_2(B)$

$\quad = \lambda_1 B + i\lambda_2 B$

$\quad = (\lambda_1 + i\lambda_2)B \quad$ for some $B \neq 0$

As $B \neq 0 \therefore \exists X \neq 0$ such that $BX \neq 0$

Therefore $A(BX) = (\lambda_1 + i\lambda_2)(BX) \quad BX \neq 0$, Therefore A has an eigenvector.

Now $n = 2^k n^1$, $k \geq 0$

For $k = 0$, we have settled the case. Hence suppose that $k \geq 1$.

Given $n = 2^k n^1$, $k \geq 1$.

Where 2^k is the highest power of 2 dividing n.

Contd...

To show that every operator has an eigenvector.

Claim: For every vector space whose dimension is not divisible by 2^k, every linear operator on V has an eigenvector. We shall prove this claim by induction on k.

If **k=1**, we have already proved the result for odd dimensions. Therefore the result is true for k=1.

We assume that the result is true for k. We shall prove that the result is true for **k+1** i.e. we consider matrices over $\mathbb{C}$ where $2^k/n$

but 2^{k+1} does not divide n.

Now let H= $\{T \in M_n (C) : T^*=T\}$ is a complex vector space of dim n(n+1)/2. Thus the highest power of 2 dividing dim H is 2^{k-1}.

By assumption every linear operator on H has an eigenvector.

Therefore by lemma 3, any pair of commuting linear operators on H has a common eigenvector.

Contd...

Now let

    $L_1 : H \text{ --->} H,$

    $L_2 : H \text{ --->} H,$

be defined by

    $L_1(B) = AB + BA^1$

    $L_2(B) = ABA^1$ $\forall B \in H$

Then L_1 & L_2 are linear operators on H, which commute. Therefore L_1 & L_2 have a common eigenvector in H.

Therefore $\exists\ O \neq B$ in H & λ & μ scalars such that

        $L_1(B) = \lambda B$ &

        $L_2(B) = \mu B$

i.e. $AB + BA^1 = \lambda B$ & $\mu B = ABA^1$

$\therefore A^2B + ABA^1 = \lambda AB.$ This means $A^2B + \mu B = \lambda AB.$

Hence $(A^2 - \lambda A + \mu)B = 0.$

Contd...

As any complex number has a complex square root, therefore any complex quadratic polynomial splits over C. This means that

$$z^2 - \lambda z + \mu = (z-\alpha)(z-\beta), \alpha, \beta \in C. \text{ Hence}$$

$$A^2 - \lambda A + \mu I = (A-\alpha I)(A-\beta I). \text{ This gives that}$$

$$(A-\alpha I)(A-\beta I)B = 0$$

$$\text{i.e. } (A-\alpha I)[(A-\beta I)B] = 0$$

Now if $(A-\beta I)B = 0$, then as $B \neq 0$, $\exists X \neq 0$ such that $BX \neq 0$

$$\text{\& then } (A-\beta I)BX = 0$$

$$A(BX) = \beta(BX) \text{ \& we are done.}$$

And If $C = (A-\beta I)B \neq 0$, then

$$(A-\alpha I)C = 0, C \neq 0 \text{ \&}$$

$$\therefore \exists X \neq 0: CX \neq 0 \text{ \&}$$

Contd...

∴ (A-αI)CX=0

∴ A(CX)=α(CX) & we are done again.

This completes the proof of the fundamental Theorem of Algebra.

[This proof given by Keith Conard is a modification of the proof given by H.Derksen]

A Visual approach:

 Here, we see a proof of the Fundamental Theorem of Algebra, by **Daniel J.Velleman**, which can be called a **"colorized"** version of **d' Alembert**'s proof of **1746.** In this proof the focus is on the use of pictures to see why the Fundamental Theorem of Algebra is true.
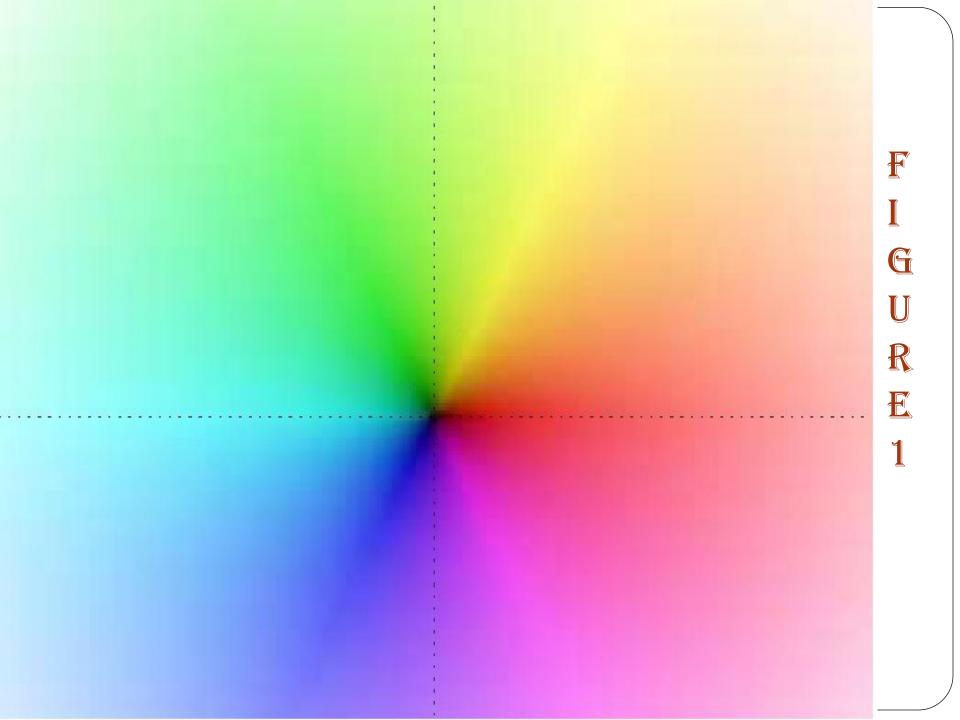
❑ Of course, if we want to use pictures to display the behavior of polynomials defined on the complex numbers, we are immediately faced with a difficulty: The complex numbers are two-dimensional, so it appears that a graph of a complex-valued function on the complex numbers will require four dimensions.

❑ Our solution to this problem will be to use colors to represent some dimensions.

❑ We begin by assigning a color to every number in the complex plane(Fig.1). The origin is colored black.

❑Points near the origin have dark colors, with the color assigned to a complex number z approaching black as z approaches 0.

❑ Points far from the origin are light, with the color of z approaching white as |z| approaches infinity.

❑ Every complex number has a different color in this picture, so a complex number can be uniquely specified by giving its color.
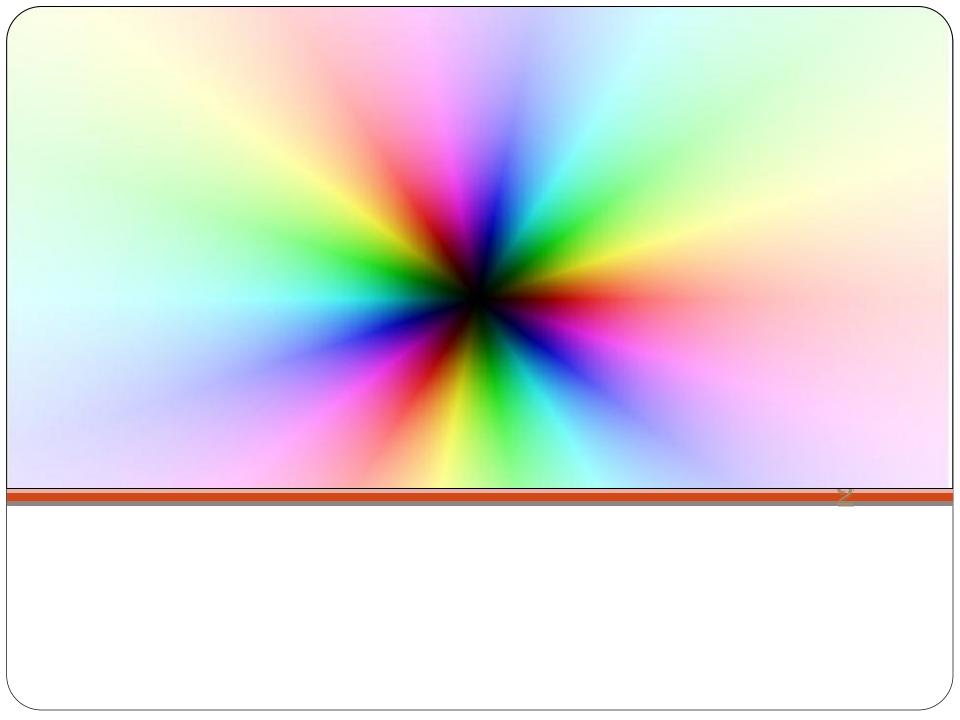
We can now use this color scheme to draw a picture of a function f : C➜C as follows:

❑We simply color each point z in the complex plane with the color corresponding to the value of f(z).

❑

❑Traveling counterclockwise around a circle centered at the origin, we go through the colors of a standard color wheel: red, yellow, green, cyan, blue, magenta, and back to red.

❑ For example, Figure 2 is a picture of the function $f(z) = z^3$. Three things are immediately evident in this picture.

❑ First, we see that the center of the picture is very dark. This is because when z is small, z^3 is very small, and therefore the color assigned to z^3 is very dark.

❑From such a picture, we can read off the value of $f(z)$, for any complex number z, by determining the color of the point z in the picture, and then consulting Figure 1 to see what complex number is represented by that color.

❑ Second, the colors fade out quickly when we move toward the outside of the picture. This is because when z is large, z^3 is very large, and therefore its color is very light.

FIGURE 1

❑ Third, and the most striking observation about the picture is that when we go counterclockwise around a circle centered at the origin, we go through the colors of the color wheel three times. This illustrates the fact that the argument of z^3 is three times the argument of z, and therefore the image of a circle centered at the origin under the cubing function wraps around the origin three times (Figure 2).

❑ We note that, from Figure 2 that every nonzero complex number has three cube roots. For example, the color assigned to the number 1 in Figure 1 is a deep red. Therefore, the three cube roots of 1 are the three points in Figure 2 that are colored this particular shade of red.

Let us now consider picture of a more complicated function (Figure 3),

$$f(z) = z^8 - 2z^7 + 2z^6 - 4z^5 + 2z^4 - 2z^3 - 5z^2 + 4z - 4.$$

Observations::

❑ The fact that the polynomial has degree eight also shows up in the picture. For large z, the z^8 term in f(z) dominates the other terms, and therefore the outer parts of the picture look similar to a picture of the function z^8: the colors begin to fade toward white as we move toward the edges of the picture.

❑ But before the colors fade out we can see that, as we go around the picture counterclockwise, the colors of the color wheel are repeated eight times.

❑ Since the color assigned to the number 0 is black, the roots

FIGURE 3

As mentioned earlier d'Alembart gave a proof of the Fundamental Theorem of Algebra in 1746. He gave a lemma which stated as:

d'Alembert's Lemma. Suppose f is a non constant polynomial, and $f(z_0) \neq 0$. Then for every $\varepsilon > 0$ there is some z such that $|z - z_0| < \varepsilon$ and $|f(z)| < |f(z_0)|$.

Let us now see a **colorized** version of the key lemma of d'Alembert's proof, which is known as Darker Neighbor Principle.

Darker Neighbor Principle. In any picture of a non constant polynomial, for any point that is not black, there is a nearby point that is darker.

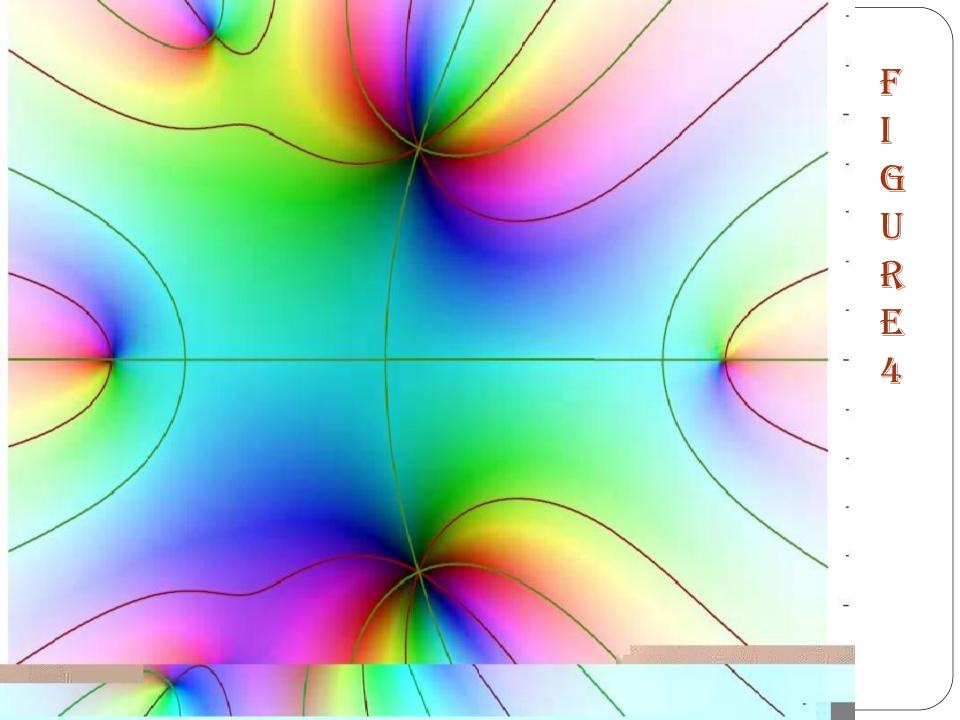"Colorized" version of d'Alembert's proof of Fundamental Theorem of Algebra of 1746.

We have already observed that, since the highest degree term of f(z) will dominate the others when z is large, the colors in the picture will fade out toward white around the outside of the picture, if R is sufficiently large. It follows that the darkest point in the picture cannot be on the boundary of S, so this darkest point will be in the interior of S. But then this point must be

black, because if it were not, then, by the Darker Neighbor Principle, some nearby point would be darker. This black point is a root of f.

This completes the proof.

ILLUSTRATION OF IDEA BEHIND GAUSS'S PROOF IN REFERENCE TO FIGURE 3::

❑ Consider separately the points where the real part of f(z) is 0 and the points where the imaginary part is 0.

❑ Now, a complex number whose imaginary part is 0 is just a real number, and in Figure 1 we can see that the color assigned to a real number is either some shade of red (if the number is positive) or some shade of cyan (if it is negative).

❑   Similarly, complex numbers whose real part is 0 are those whose color is

some shade of either yellow-green or magenta-blue.

❑ <u>Figure 4</u> is a copy of Figure 3 in which all of these points have been marked. The red curves in Figure 4 are the points where the real part of f(z) is 0, and the green curves are the points where the imaginary part is 0.

❑ As we observed earlier, going around the border of Figure 3, the color wheel cycle of colors is repeated eight times. Each cycle includes all four of the colors red, yellow-green, cyan, and magenta blue, in order, and so along the border of Figure 4 there are 32 ends of curves, alternating red and green.

❑ If we start at any one of these curve ends and follow the curve into the picture, we will emerge at another curve end of the same color.

FIGURE 4

Thank You