Sets, Relations & Functions

Problem Set-1

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1. Justiy which of the following are true/false:
   (a) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$
   (b) If $x \in A$ and $A \subseteq B$ then $x \in B$
   (c) If $A \subseteq B$ and $x \notin B$ then $x \notin A$
   (d) $A \cap B = A \cap C \Rightarrow B = C$
   (e) $A \cup B = A \cup C \Rightarrow B = C$
   (f) $(A \cap B) \cup (A - B) = A$
   (g) $A \cap (B \cup A)^c = A \cap B$
   (h) Every subset of an infinite set is infinite.

2. If $A = \{x \in \mathbb{C} : x^2 = 1\}$ and B=$\{x \in \mathbb{C} : x^4 = 1\}$ find $A - B$.

3. If $A = \{2, 5, 10, 17, ..., 101\}$ find n(P(A))

4. If $n(A) = 110, n(B) = 300, n(A - B) = 50$ then find $n(A \cup B)$

5. If $n(U) = 2000, n(A) = 1720, n(B) = 1450$ then find the least value of $n(A \cap B)$

6. Let $n(U) = 700, n(A) = 180, n(B) = 275$ and $n(A \cap B) = 95$ then show that $n(A^c \cap B^c) = 300$

7. Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find m and n.

8. In a town of 10000 families it was found that 40% families buy newspaper A, 20% buy newspaper B and 10% families buy newspaper C, 5% buy both A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three newspapers, then find the number of families which buy (i) only A, (ii) only B, (iii) only C , (iv) none.

9. In a group of 50 people, 30 play cricket, 256 play football and 32 play hockey. Assume that each person of the group plays at least one of the three games. If 15 people play both cricket and football, 11 play both football and hockey and 18 play both cricket and hockey, then how many people (i)play all the three games, (ii) play football only, (iii) do not play hockey, (iv) do not play football?

10. Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. Is this data correct?

11. If $A = \{(x, y) : x + 2y = 3\}$ and $B = \{(x, y) : 3x + 2y = 5\}$, find $A \cap B$.

12. If $A = \{(x, y) : x^2 + y^2 \leq 4\}$ and $B = \{(x, y) : x^2 + y^2 \geq a\}$. Then for what values of a will $A \cap B$ be non-empty?

13. Let $M = \{(x, y) : y \geq x^2\}$ and $N = \{(x, y) : x^2 + (y - a)^2 \leq 1\}$. Then for what values of a, $M \cap N$ will be equal to N?

14. Classify the following relations as reflexive, symmetric, transitive or their combinations:
   (a) $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on the set $A = \{1, 2, 3\}$
   (b) $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ on the set $A = \{3, 6, 9, 12\}$
   (c) Let R be defined on $\mathbb{Z}$ by $(x, y) \in R \Leftrightarrow x - y = 10$
   (d) Let R be defined on $\mathbb{Z}$ by $(x, y) \in R \Leftrightarrow x - y = 0 \;\; or \;\; 10$
   (e) Let R be defined on $\mathbb{Z}$ by $(x, y) \in R \Leftrightarrow |x - y| = 0 \;\; or \;\; 10$
   (f) Let R be defined on $\mathbb{Z}$ by $(x, y) \in R \Leftrightarrow x - y$ is divisible by n
   (g) Let R be a relation on the set $\mathbb{N}$ of natural numbers defined by $(n, m) \in R \Leftrightarrow n|m$.
   (h) Let R be a relation on $\mathbb{N}$ defined by $(x, y) \in R \Leftrightarrow x^2 - 4xy + 3y^2 = 0$

(i) Let W be the set of words in a particular English dictionary. Define R={ (x,y)∈ W× W: x and y have at least one letter in common }

(j) Let A be the set of all straight lines which lie on the plane of this paper. Let relation R on A be defined by $(x, y) \in \mathbb{R} \Leftrightarrow$ x ⊥ y

(k) Let A be the set of all straight lines which lie on the plane of this paper. Let relation R on A be defined by $(x, y) \in \mathbb{R} \Leftrightarrow x \parallel y$

(l) Let R be a relation on $\mathbb{R}$ defined by $(x, y) \in \mathbb{R} \Leftrightarrow |x - y| \geq 0$

(m) Relation R defined on $\mathbb{Z}$ by $(x, y) \in \mathbb{R} \Leftrightarrow xy \geq 0$

15. Give example of a relation which is

    (a) reflexive, symmetric but not transitive.

    (b) reflexive, transitive but not symmetric.

    (c) symmetric, transitive but not reflexive.

    (d) reflexive but neither transitive nor symmetric.

    (e) symmetric but neither reflexive nor transitive.

    (f) transitive but neither reflexive nor symmetric.

16. Let R be a relation on $\mathbb{N}$ defined by $R = \{(x, y) : x + 2y = 8\}$. Then find the range of R

17. Given the relation $R = \{(1, 2), (2, 3)\}$ on the set $A = \{1, 2, 3\}$, what is the minimum number of ordered pairs required to make it an equivalence relation?

18. Find the range of the function $f(x) = {}^{7-x}P_{x-3}$.

19. Which of the following are true for relations R and S on a set A?

    (a) If R and S are transitive then so is $R \cup S$

    (b) If R and S are transitive then so is $R \cap S$

    (c) If R and S are reflexive then so is $R \cup S$

    (d) If R and S are symmetric then so is $R \cup S$

20. How many relations can be there on a set having n elements?

21. How many relations can be there from a set having m elements to a set having n elements?

22. How many functions can be there on a set having n elements?

23. How many functions can be there from a set having m elements to a set having n elements?

24. How many one one functions can be defined on a set having n elements?

25. How many one-one functions can be there from a set having m elements to a set having n elements?

26. How many binary operations can be defined on a set having n elements?

27. Let $R = \{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 \leq 25\}$. And $R' = \{(x, y) : x, y \in \mathbb{R}, y \geq \frac{4}{9}x^2\}$ Then

    (a) Domain $R \cap R' = [-3, 3]$

    (b) Range $R \cap R' \supseteq [0, 4]$

    (c) Range $R \cap R' = [0, 5]$

    (d) all of these

28. Find the domain and range of the following functions :

    (a) $f(x) = x + 1$ $\qquad g(x) = \dfrac{x^2 - 1}{x - 1}$

    (b) $f(x) = \sqrt{x^2 - 9}$ $\qquad g(x) = \sqrt{9 - x^2}$

    (c) $f(x) = \dfrac{1}{\sqrt{x - 1}}$ $\qquad g(x) = \dfrac{1}{\sqrt{1 - x}}$

    (d) $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$ $\qquad g(x) = \dfrac{1}{\sqrt{1 - x^2}}$

    (e) $f(x) = \dfrac{|x - 3|}{x - 3}$ $\qquad g(x) = \dfrac{|x| - 3}{x - 3}$

    (f) $f(x) = \dfrac{1}{\sqrt{|x| + 1}}$ $\qquad g(x) = \dfrac{1}{\sqrt{|x + 1|}}$

    (g) $f(x) = \dfrac{1}{1 + \lfloor x \rfloor}$ $\qquad g(x) = \dfrac{1}{\sqrt{1 + \lfloor x \rfloor}}$ where $\lfloor x \rfloor$ is the floor function (greatest integer less than or equal to x)

    (h) $f(x) = \dfrac{1}{x - \lfloor x \rfloor}$ $\qquad g(x) = \dfrac{1}{x - |x|}$

    (i) $f(x) = \dfrac{\cos^{-1} x}{\lfloor x \rfloor}$

29. Find the domains of the following functions :

    (a) $f(x) = \dfrac{3}{4 - x^2} + \dfrac{1}{\sqrt{x + 2}}$

(b) $f(x) = \dfrac{3}{4-x^2} + \log(x^3 - x)$

(c) $f(x) = \sin^{-1}(\log_3 \dfrac{x}{3})$

(d) $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$

(e) $f(x) = \sqrt{2\sin x - 1}$

(f) $f(x) = \sqrt{\log_{0.4} \dfrac{x-1}{x+5}} \times \dfrac{1}{x^2-36}$

(g) $f(x) = \dfrac{\sqrt{-\log_{0.3}(x-1)}}{\sqrt{-x^2+2x+8}}$

(h) $f(x) = \dfrac{1}{x} + 2^{\sin^{-1} x} + \dfrac{1}{\sqrt{x-2}}$

30. $f(x) = \sqrt{\dfrac{\cos x - \frac{1}{2}}{6 + 35 - 6x^2}}$

31. Let $f(x) = x^2 + 4, g(x) = \dfrac{1}{\sqrt{x-1}}$. Then

   (a) dom $(f+g) = (0, \infty) - (0, 1]$

   (b) Range $f \cap$ Range $g = [4, \infty)$

   (c) Range $g = [4.\infty)$

   (d) All of these.

32. Let $f(x) = \dfrac{x}{1+x^2}, g(x) = \dfrac{e^{-x}}{1+\lfloor x \rfloor}$. Then

   (a) $\text{dom}(f+g) = \mathbb{R} - [-2, 0)$

   (b) $\text{dom}(f+g) = \mathbb{R} - [-1, 0)$

   (c) Range $f \cap$ Range $g = [-\frac{1}{2}, \frac{1}{2}] - \{0\}$

   (d) Both (b) and (c)

33. Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 3n$ and
$$g(n) = \begin{cases} \dfrac{n}{3} & , n \text{ is a multiple of } 3 \\ 0 & , \text{otherwise} \end{cases} \quad \forall n \in \mathbb{Z}$$
Show that $gof = I_{\mathbb{Z}}$ but $fog \neq I_{\mathbb{Z}}$

34. If $f : A \to B$ and $g : B \to A$ be two one-one (injective) functions then

   (a) gof is injective

   (b) fog is one-one.

   (c) gof and fog are onto.

   (d) Both (a) and (b)

35. If $f : A \to B$, $g : B \to A$ be two functions such that $gof = I_A$ then

   (a) g is one-one and f is onto.

   (b) f is one-one and g is onto.

   (c) $f = g = I_A$

36. Let $f(x) = \sin\left(\dfrac{\pi}{x}\right)$ and $D_+ = \{x : f(x) > 0\}$. Then D_+ contains

   (a) $(\frac{1}{3}, \frac{1}{2})$

   (b) $(\frac{1}{5}, \frac{1}{4})$

   (c) $(-1, -2)$

   (d) all of these

   Also show that D_+ contains $(1, \infty)$ and $(\frac{1}{2n+1}, \frac{1}{2n}), n \in \mathbb{Z} - \{0\}$

37. Construct a bijective function from $\mathbb{N}$, the set of natural numbers, to $\mathbb{E}$, the set of even numbers.

38. Construct a bijective function from $\mathbb{N}$ to $\mathbb{O}$, the set of odd numbers.

39. Construct a bijective function from $\mathbb{N}$ to $\mathbb{Z}$, the set of integers.

40. Construct a bijective function from $\mathbb{N}$ to $\mathbb{Q}$, the set of rational numbers.

41. Construct a bijective function from $\mathbb{Z}$ to $\mathbb{Q}$.

42. Construct a bijective function from any closed interval $[a, b]$ to any other closed interval $[c, d]$.

43. Construct a bijective function from $(0, 1)$ to $\mathbb{R}$, the set of real numbers.

44. Draw the graphs of :

   (a) $f(x) = \sin x, g(x) = 2\sin x, h(x) = \sin 2x$

   (b) $f(x) = 2\cos^2 \dfrac{x}{2}, g(x) = 2\sin \dfrac{x}{2} \cos \dfrac{x}{2}$

   (c) $f(x) = \max\{x, x^2\}$

   (d) $f(x) = \min\{x, x^2\}$

   (e) $f(x) = |x+1|, g(x) = |x| + 1$

   (f) $f(x) = x + |x|, g(x) = x - |x|$

   (g) $f(x) = \dfrac{|x|}{x}$

   (h) $f(x) = \lfloor x \rfloor, g(x) = \lfloor x \rfloor + 1$

   (i) $f(x) = 2\lfloor x \rfloor, g(x) = \lfloor 2x \rfloor$

   (j) $f(x) = \dfrac{1}{x}, g(x) = \dfrac{1}{x-1}$

   (k) $f(x) = \sqrt{x}, g(x) = \sqrt{-x}$

   (l) $f(x) = -\sqrt{x}, g(x) = -\sqrt{-x}$

   (m) $f(x) = x + x^2, g(x) = x - x^2$

   (n) $f(x) = ax^2 + bx + c, a > 0$

   (o) $f(x) = ax^2 + bx + c, a < 0$