A New Quadrature Rule For Numerical Integration Using Small Circles

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This article deals with the numerical integration techniques in new way. We develop a new quadrature rule for numerical integration. Apart from all existing rule which uses vertical stripes in different ways to find the area under the curve. In this study, we use small circles to approximate it. Though the approximation in this rule is very rough, however the formulae consist the mathematical constant π. We also evaluate the error of this formula.

Quadrature is historical term that means calculating area. Quadrature is very important topic in mathematical analysis. Mathematicians of ancient Greece, according to the Pythagorean doctrine, understood determination of area of a figure as the process of geometrically constructing a square having the same area (*squaring*).Thus the named quadrature for this process. The Greek geometers were not always successful (see quadrature of the circle), but they did carry out quadrature's of some figures whose sides were not simply line segments, such as the Lunes of Hippocrates and the quadrature of the parabola. By Greek tradition, these constructions had to be performed using only a compass and straightedge. [1]

INTEGRATION

What is integration? Historically, integration is defined as inverse of differentiation, but later it is completely defined more rigorously as a new notion by mathematician Bernard Riemann as 'area under the curve', which is known as Riemann integral. Riemann integral is a method in which the graph of a curve is spitted into small stripes (as given in the **Figure 1**) and the length Δx(difference between consecutive x-axis co-ordinates) is tends to 0, so it gives a very accurate area of curve. Starting with a function $f(x)$ on $[a, b]$, we partition the domain into small subintervals. On each subinterval $[x_{k-1,} x_k]$, we pick some point $c \in [x_{k-1,} x_k]$ and use the y-value $f(c_k)$ as an approximation for $f(x)$ on $[x_{k-1,} x_k]$. Graphically speaking, the result is a row of thin rectangles constructed to approximate the area between $f(x)$ and the x-axis. The area of each rectangle is $f(c_k)(x_k - x_{k-1})$, and so the total area of all of the rectangles is given by the Riemann sum.

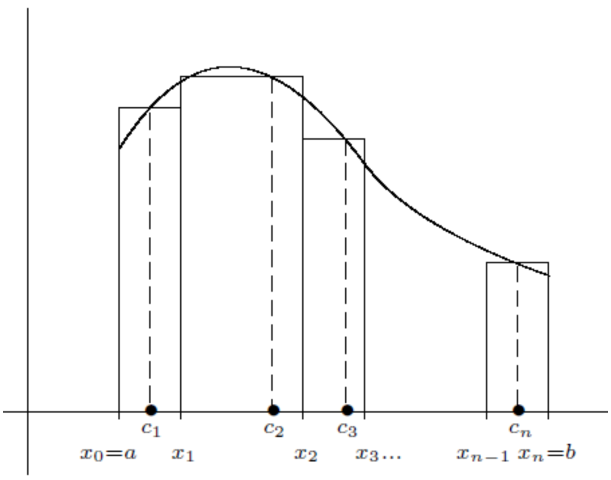

Figure 1

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NUMERICAL INTEGRATION

Numerical integration is study of how the numerical value of an integrand can be found. The beginnings of this subject are to be sought in antiquity. A fine example of ancient numerical integration, but that one is entirely in the spirit of the present volume, is the Greek quadrature of the circle by means of inscribed and circumscribed regular polygons. This process led Archimedes to an upper and lower bound for the value of π. Over the centuries, particularly since the sixteenth century, many methods of numerical integration have been devised. These include the use of fundamental theorem of integral calculus, infinite series, functional relationships, differential equations, and transforms.

There are several reasons for carrying out numerical integration. As we have seen that numerical integration used when analytical techniques fails. The integrand $f(x)$ may be known only at certain points, such as obtained by sampling. Some embedded systems and other computer applications may need numerical integration for this reason. A formula for the integrand may be known, but it may be difficult or impossible to find an antiderivative that is an elementary function. An example of such an integrand is $f(x) = e^{-x^2}$, the antiderivative of which (the error function, times a constant) cannot be written in elementary form.

It may be possible to find an antiderivative symbolically, but it may be easier to compute a numerical approximation than to compute the antiderivative. That may be the case if the antiderivative is given as an infinite series or product, or if its evaluation requires a special function that is not available [2]. There are many numerical integration techniques but commonly used are Trapezoid rule and Simpson rule. Below we emphasis these both rule.

Trapezoid Rule

We have to approximate the definite integral $\int_a^b f(x)dx$ using trapezoid rule. We assume that $f(x)$ is continuous on $[a, b]$ and we divide $[a, b]$ into n subintervals of equal length.
The x_n part of the curve can be calculated as $x_n = a + n\Delta x$ and also we get $\Delta x = \frac{b-a}{n}$.
In trapezoid rule, the area of each column is calculated considering it as a trapezoid.(**Figure2)**

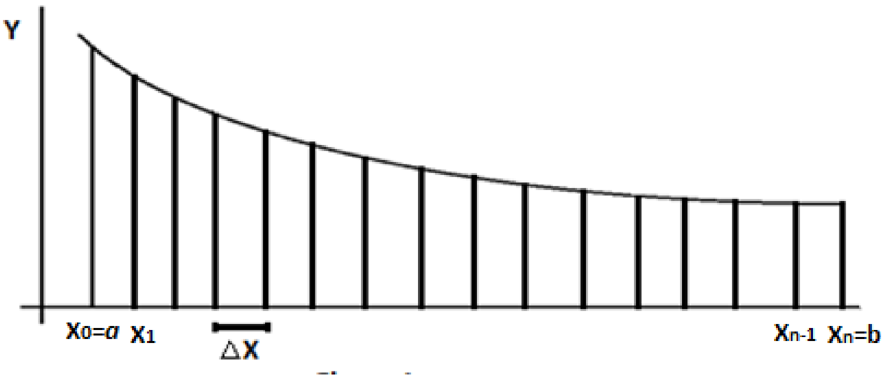

Figure 2

The area of a trapezoid is obtained by adding the area of a rectangle and a triangle **(Figure 3)**.

Area $= y_0\Delta x + \frac{1}{2}(y_1 - y_0)\Delta x = \frac{(y_1+y_0)\Delta x}{2}$

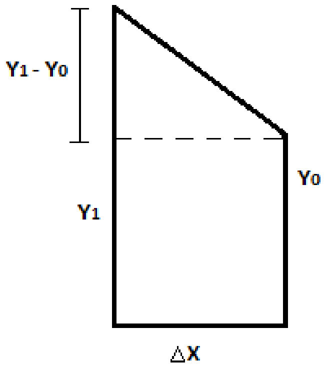

Figure 3

By adding the area of the n trapezoids, we obtain the approximation

$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(y_0 + 2(y_1 + y_2 + y_3 \dots + y_{n-1}) + y_n)$

This is formula for trapezoid rule.

Simpson Rule

Simpson's rule is a numerical method that approximates the value of a definite integral by using quadratic polynomials.

Let's first derive a formula for the area under a parabola of equation $y = ax^2 + bx + c$ passing through the three points : $(-h, y_0)$, $(0, y_1)$ and (h, y_2).**(Figure 4)**

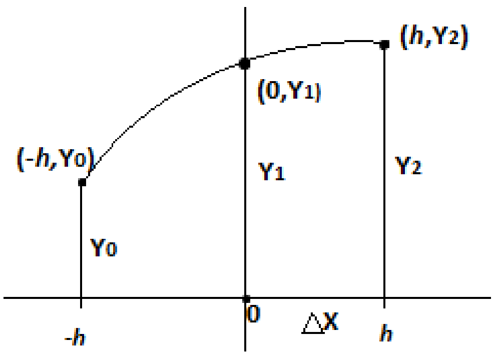

Figure 4

Area $= \int_{-h}^{h}(ax^2 + bx + c)dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h} = \frac{h(2ah^2+6c)}{3}$

Since the points$(-h, y_0)$, $(0, y_1)$ and (h, y_2) are on parabola, they satisfy $y = ax^2 + bx + c$

Therefore,

$$y_0 = ah^2 - bh + c$$

$$y_1 = c$$

$$y_2 = ah^2 + bh + c$$

Observe,

$$y_0 + 4y_1 + y_2 = (ah^2 - bh + c) + 4c + ah^2 + bh + c = 2ah^2 + 6c$$

Area $= \frac{h(y_0+4y_1+y_2)}{3} = \frac{\Delta x(y_0+4y_1+y_2)}{3}$

We assume that $f(x)$ is continuous on $[a, b]$ and we divide $[a, b]$ into an even number n of subintervals of equal length. **(Figure 5)**

The x_n part of the curve can be calculated as $x_n = a + n\Delta x$ and also we get $\Delta x = \frac{b-a}{n}$.

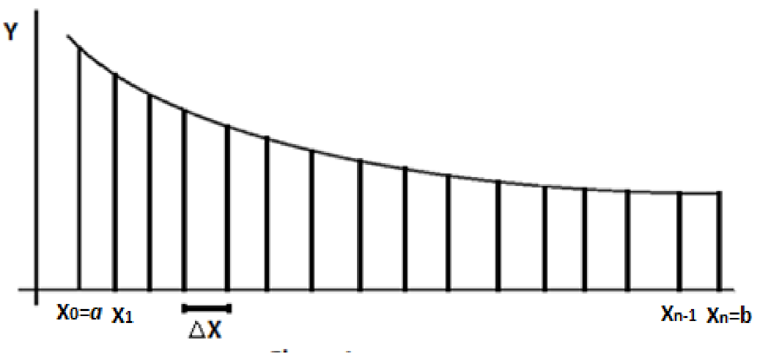

Figure 5

We can estimate the integral by adding the areas under the parabolic arcs through three successive points.

$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 \ldots + 4y_{n-1} + y_n)$

This is formula for Simpson rule.

Our article approaches the idea of numerical integration using small circles. We use different properties of circles and graphs to derive the formulae for numerical integration. Finally, we use Taylor's theorem to compute the error in this formula.

DERIVATION OF NUMERICAL INTEGRATION FORMULA USING SMALL CIRCLES

Let the $f(x)$ from $[a, b] \to \mathbb{R}$ be the real valued function and $f(x)$ is continuous on $[a, b]$. Now, we apply our procedure of new quadrature rule to approximate integral of $\int_a^b f(x)dx$.**(Figure 6)**

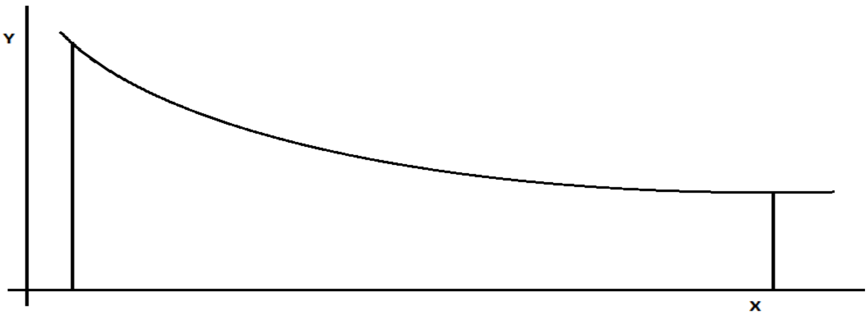

Figure6. Curve $f(x)$

First we divide the area under curve $f(x)$ in small stripes and call this as partition and denote each partition by $P_1, P_2, P_3 \dots P_n$ which corresponds to points $x_1, x_2, x_3 \dots x_n = b$. Then we fill each partition with small circles, with same radius r.**(Figure 7)**

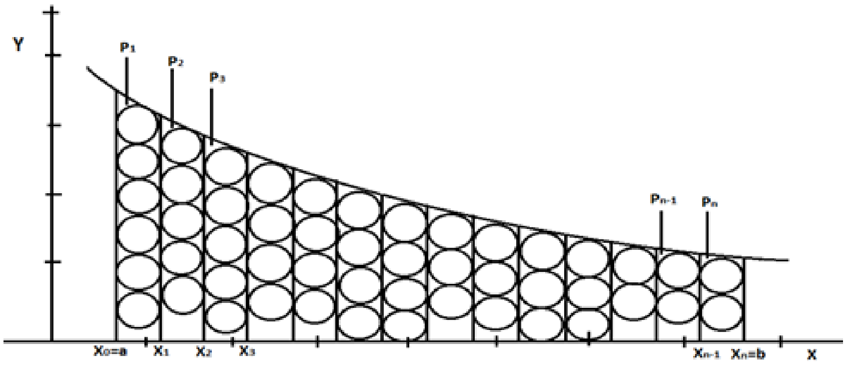

Figure 7. Graph of $f(x)$ filled with circles

Each partition has a width Δx and let the number of partitions be n. So, by the above ideas we get $\Delta x = 2r$ and we have $\Delta x = \frac{b-a}{n}$ **(Figure 8)**. So, we get,

$$r = \frac{b-a}{2n} \quad \textbf{Equation 1.}$$

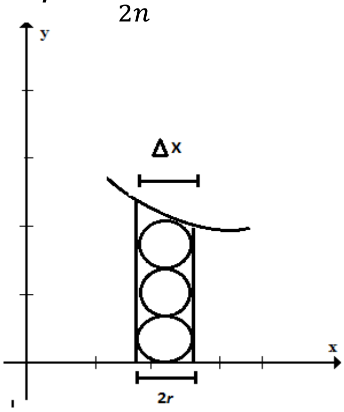

Figure 8

Since, each x_i(i =0 to n) corresponds to respective $f(x_i)$. Now we calculate the average height of each partition. So average height = $\frac{f(x_{i-1}) + f(x_i)}{2}$ **(Figure 9)**

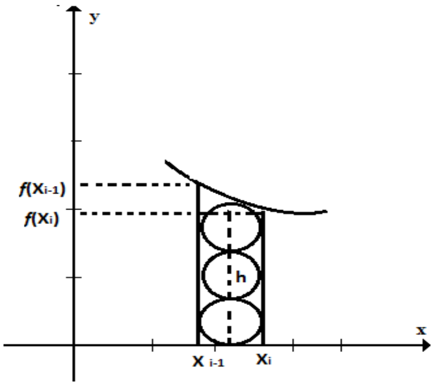

Figure 9

Let C_i denotes the number of circles in the corresponds partition P_i .So we estimate the approximate number of circles in a partition

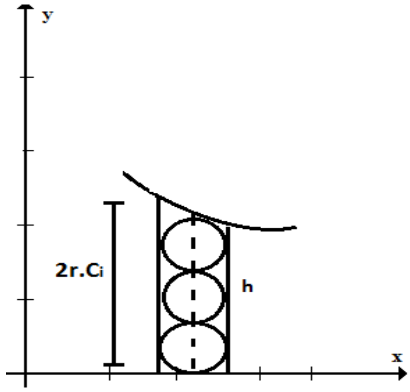

Figure 10

$$2rC_i = \frac{f(x_{i-1}) + f(x_i)}{2} \Rightarrow C_i = \frac{f(x_{i-1}) + f(x_i)}{4r} \quad \textbf{(Figure 10)}$$

This is equation for number of circles in i^{th}partition.

To calculate the x_i value, we use the formula $x_i = a + \Delta x i = a + \left(\frac{b-a}{n}\right) i$,since i^{th} position is sum of shifting from initial point to Δx times i^{th} position.

So in order to calculate $\int_a^b f(x)dx$ i.e. area under the curve to find the total number of circles in a given curve. Now, we calculate total number of circles in a given curve $f(x)$.

$$\sum_{i=1}^{n} C_i = \frac{f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k)}{4r} \qquad \textbf{Equation 2}$$

Since, we know that area of circle is πr^2.We use this fact to calculate the area under the curve. Let $Ar(P_i)$ denotes the area in i^{th} partition. So, the total area of curve will be $\pi r^2 \times C_i$ i.e. product of area of each circle to number of circle in each partition.

Total area = $\sum_{i=1}^{n} Ar(P_i) = \pi r^2 \sum_{i=1}^{n} C_i$

Then using **Equation 1** and **2** we get,

$$\frac{\pi(b-a)(f(a) + f(b) + 2\sum_{k=1}^{n-1} f(x_k))}{8n}$$

Using the notion $x_i = a + \left(\frac{b-a}{n}\right) i$,we get $\sum_{k=1}^{n-1} f(x_k) = \frac{(n-1)(a+b)}{2}$ then, the above equation reduces to

$$\frac{\pi(b-a)(f(a)+f(b)+(n-1)(a+b))}{8n} \qquad \textbf{Equation 3}$$

This is formula for numerical integration formula using small circles.

Example : Let $f(x) = x$ and we have to evaluate $\int_0^1 x\, dx$

Now we use our numerical integration formula **Equation 3**, for n =2, so we get

$$\frac{\pi(1-0)(f(1)+f(0)+2\ (0+1))}{8\times 2} = \frac{\pi}{8} = 0.3926$$

The original integration $\int_0^1 x\, dx = \ 0.5$

Error estimate = 0.5 - 0.3926 = 0.1074

ERROR ANALYSIS

It is quite visible that the vacant space between circles is main cause of error. Also, sometimes there are some functions in which we cannot draw full circles at edges **(Figure 11)**.

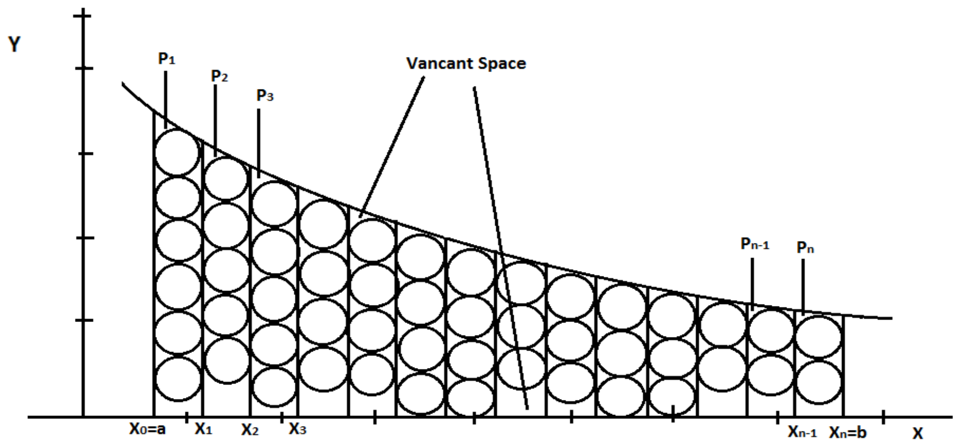

Figure 11

There is another conception that arises, if we increase the value of n then we get good approximation, but it is not always correct, since it depends on the type of function. In some function, whose graph is convenient to draw equal and full circles we get better approximation.

Now we compute the error in this formulae i.e.

$$\int_a^b f(x)dx - \frac{\pi(b-a)\left(f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k)\right)}{8n}$$

Let we assume that $f(x)$and its first two derivatives are continuous on the interval $[a, b]$, and $M = \max x \in [a, b] \left|f''(x)\right|$.We use the notion of Taylor theorem to compute the error.

$f(x) = f(a_{1/2}) + (x - a_{1/2})f'(a_{1/2}) + \frac{(x-a_{1/2})^2 f''(c_1)}{2!}$, where $a_{1/2} = \frac{a+b}{2}$ and $c_1 \in [a_{1/2}, x]$

Now we compute $\int_a^b f(x)dx$,

$$\int_a^b f(x)dx = \int_a^b \left(f(a_{1/2}) + (x - a_{1/2})f'(a_{1/2}) + \frac{(x-a)^2 f''(c_1)}{2!}\right)dx$$

$$= f(a_{1/2})(b-a) + \frac{f'(a_{1/2})}{2}\left[\frac{x^2}{2} - a_{1/2}x\right]_a^b + \int_a^b \frac{(x-a_{1/2})^2 f''(c_1)}{2!}$$

$$= f(a_{1/2})(b-a) + \int_a^b \frac{(x-a_{1/2})^2 f''(c_1)}{2!}$$

Then an upper bound for $\int_a^b f(x)dx$ is,

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

$$\leq \int_a^b \left|f(a_{1/2})(b-a) + \int_a^b \frac{(x-a_{1/2})^2 f''(c_1)}{2!}\right| dx$$

$$\leq \int_a^b |f(a_{1/2})(b-a)|dx + \int_a^b \left|\frac{(x-a_{1/2})^2 f''(c_1)}{2!}\right| dx$$

Since we have $M = \max x \in [a,b]\,|f''(x)|$, now we integrate it two times from a to b with respect to x, we get $M(b-a)^2 = \max x \in [a,b]\,|f(x)|$, so $|f(a_{1/2})| = M(b-a)^2$, since $a_{1/2} \in [a,b]$,

Now simplifying above equation we get

$$= M(b-a)^4 + \frac{M(b-a)^3}{24} \qquad \textbf{Equation 4}$$

Again we use the notion of Taylor theorem to compute $f(a) + f(b)$

$f(a) = f(a_{1/2}) + (a - a_{1/2})f'(a_{1/2}) + \frac{(a-a_{1/2})^2 f''(c_2)}{2!}$, where $c_2 \in [a_{1/2}, a]$

$f(b) = f(a_{1/2}) + (b - a_{1/2})f'(a_{1/2}) + \frac{(b-a_{1/2})^2 f''(c_3)}{2!}$, where $c_3 \in [a_{1/2}, b]$

$$f(a) + f(b) = 2f(a_{1/2}) + \frac{(b-a)^2 f''(c_2)}{8} + \frac{(b-a)^2 f''(c_3)}{8}$$

Now we evaluate $\left|\frac{\pi(b-a)\left(f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k)\right)}{8n}\right|$

$$\leq \frac{\pi(b-a)}{8n}\left(2|f(a_{1/2})| + \frac{(b-a)^2|f''(c_2)|}{8} + \frac{(b-a)^2|f''(c_3)|}{8} + 2\sum_{k=1}^{n-1}|f(x_k)|\right)$$

$$\leq \frac{\pi(b-a)}{8n}\left(2M(b-a)^2 + \frac{(b-a)^2 M}{8} + \frac{(b-a)^2 M}{8} + 2M(n-1)(b-a)^2\right)$$

$$\leq \frac{M\pi(b-a)^3}{4n} + \frac{M\pi(b-a)^3}{32n} + \frac{M\pi(n-1)(b-a)^3}{4n} \qquad \textbf{Equation 5}$$

Now we compute the global error using the **Equation 4** and **Equation 5**

$$\left|\int_a^b f(x)dx - \frac{\pi(b-a)\left(f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k)\right)}{8n}\right| \leq \left|\int_a^b f(x)dx\right| + \left|\frac{\pi(b-a)\left(f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k)\right)}{8n}\right|$$

$$\leq M(b-a)^4 + \frac{M(b-a)^3}{24} + \frac{M\pi(b-a)^3}{4n} + \frac{M\pi(b-a)^3}{32n} + \frac{M\pi(n-1)(b-a)^3}{4n}$$

Therefore equation for error is

$$M(b-a)^4 + \frac{M(b-a)^3}{24} + \frac{M\pi(b-a)^3}{4n} + \frac{M\pi(b-a)^3}{32n} + \frac{M\pi(n-1)(b-a)^3}{4n} \quad \textbf{Equation 6}$$

CONCLUSION

In this article, we use a new way to approach quadrature rule, though the error is very large, but this idea can be extend by using group of small circles in each partition such that the radius of circles tends to very small magnitude (a point circle), to get more accurate area under the curve**(Figure 12)**.

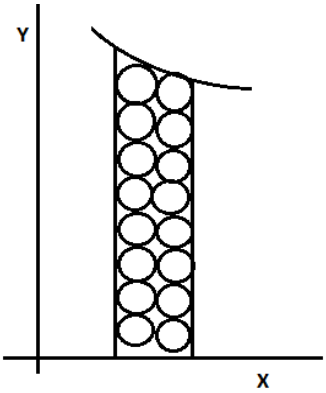

Figure 12

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