Game Theory: Can there be a theory to play a game?

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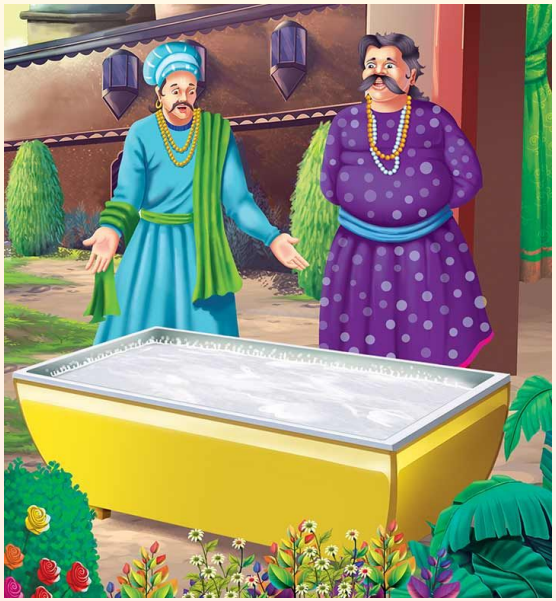
Game Theory : What is it all about?

Akbar and Birbal : The Bathtub with full of Milk!



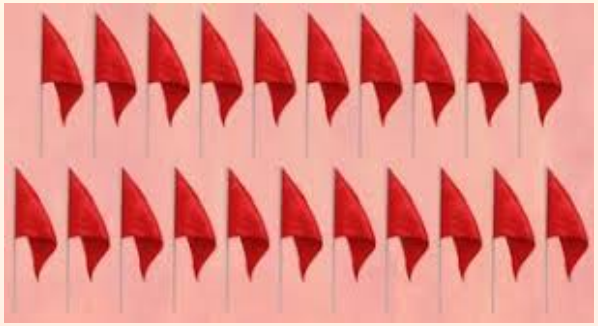
- Akbar wanted to take bath with milk!
- Ordered all the 100 ministers to bring a bucket of milk to fill the bathtub!
- Each thought that the remaining 99 would bring milk!
- No difference if he brought water!
- The bathtub was filled with water!

The Bottomline is-

The best way for everyone was to bring water instead of milk!

The Survivor Thailand

- There were 21 flags and 2 players, who alternated in taking turns to remove some flags.
- At each turn, the player had to remove 1, 2, or 3 flags;
- This was the player's choice at each move.
- The player who removed the last flag (whether as the sole remaining flag or one of the last surviving set of 2 or 3 flags) was the winner.



The Theory to Win $ 30,000,000!!!

- If player 1 leaves player 2 with four flags, player 2 must remove 1, 2, or 3, and then player 1 can take the rest and win.
- To make sure that player 1 leaves player 2 with four flags, player 1 must leave player 2 facing eight flags on the immediately preceding turn.
- The logical sequence then is to leave 12, 16, and 20 on previous turns.
- Therefore, starting with 21 flags, player 1 should remove one and proceed to take four minus whatever player 2 takes at the immediately preceding turn.

The Generalized Beauty Contest: Herve Moulin

The Problem

- Choose 10 students in the class and give them blank cards.
- Each writes her name on the card and a number between 0 and 100;
- Collect the cards and average the numbers on them.
- The student whose choice is closest to half of the average is the winner.

The Solution

- The average can never exceed 100.
- Half of the average can never exceed 50.
- Any choice above 50 is dominated by 50.
- Then the average can never exceed 50...Then half should be around 25.

The Bottomline is-

The best number for everyone should be 0— The Nash Equilibrium!!!

Game: The Informal way

- A Game is a situation with multiple decision makers —Players
- Each player endowed with a set of actions —Strategies
- When each player adopts a Strategy...It results in an OUTCOME—Payoff...

Examples from the Society: The Battle of Sexes

Back

Definition

- A couple deciding how to spend the evening.
- Wife would like to go for a movie.
- Husband would like to go for a cricket match.
- Both however want to spend the time together
- Scope for strategic interaction!

		Player II	
		y	
Player I	x	A: Movie	Cricket
		Movie	$(2,\ 1)$	$(0,\ 0)$
		Cricket	$(0,\ 0)$	$(1,\ 2)$

Table: Payoff Matrix A : The Nash Equilibrium?

The Corruption Game

Back

Definition

- A Businessman wants a permit from the Govt. official.
- He must choose between Honest behaviour or to Bribe the officer (grease the wheel of Bureaucracy).
- The Officer chooses either to be Honest or take the bribe.
- What will they do? No coordination –> Chance of prosecution!

Govt. Officer (Player II) y

Businessman(Player I) x	A	Honest	Corrupt
	Honest	$(3, \ 1)$	$(0, \ -2)$
	Corrupt	$(-2, \ 0)$	$(1, \ 3)$

Table: Payoff Matrix A : The Nash Equilibrium?

The Wearing Mask Game

Back

- Two persons, X and Y, approach each other within 1 meter distance.
- They have the choices of wearing masks or not!!!
- If X wears a mask, Y is safe.
- If Y wears a mask, X is safe.
- Will they wear a mask to be termed as socially responsible? Do they have incentive for that?

		Y	
X	A	Wearing a Mask	Refuse to wear a Mask
	Wearing a Mask	$(10, \ 10)$	$(0, \ 10)$
	Refuse to Wear a Mask	$(10, \ 0)$	$(0, \ 0)$

Table: Payoff Matrix A : The Nash Equilibrium?

The Prisoner's Dilemma

Back

- Two convicts are put to two cells.
- They either both confess of their guilty or do not confess.
- If both confess their jail term will be 3 years each! If both don't confess then 1 year jail!
- If one confesses and the other does not, the former considered govt. witness–freed.
- The latter gets the highest punishment!

B

A		Confess	Not Confess
	Confess	$(3, \ 3)$	$(0, \ 9)$
	Not Confess	$(9, \ 0)$	$(1, \ 1)$

Examples of the Prisoner's Dilemma

- The Advertisement Game! War between RJ Reynolds and Philip Morris in the US before 1970!
- The Environment Pollution Game!
- The Arms Racing Games!

The Bottomline is-

- Two players have strategies "Cooperate" or "Defect"!
- What will be their best response to each other's strategies?

The Game in the story of Akbar and Birbal

Back

- Let us take two ministers instead of 100: “Bring Milk” or “Bring Water"!
- Payoff for bringing Milk when both bring Milk $= s$. (sustainable)
- Payoff for bringing Milk when the other brings Water $= v$. (victim)
- Payoff for bringing Water when the other brings Milk $= r$. (Rogue)
- Payoff for bringing Water when both bring Water $= t$. (Commons Tragedy Payoff)
- $$r > s > t > v.$$

B

A	A	Bring Milk	Bring Water
	Bring Milk	$(s, \quad s)$	$(v, \quad r)$
	Bring Water	$(r, \quad v)$	$(t, \quad t)$

Game Theory

Definition

Game theory models strategic behaviour by *intelligent* and *rational* agents who understand that their actions affect the actions of other agents.

Definition

A game consists of

- a set of players;
- a set of strategies for each player;
- the payoffs to each player for every possible list of strategy choices by the players.

Two Basic Assumptions of Game Theory

- Players be **rational** - If the player makes decision in pursuit of his own **Objectives**: to maximize the expected value of his own payoff (Bernoulli, 1738).
  - Maximization is done by using some **utility** function.
  - Utility function exists, (von Newmann and Morgenstern, 1947).

- Players be **intelligent** - know everything that we know about the game and can make any inference of the theory that we can make.
  - If we develop a theory and we believe that it is correct then the assumption is that each player in the game will also understand this theory and its predictions.
  - If a theory predicts that some individuals will be systematically fooled or led into making costly mistakes then this theory will tend to loose its validity when those players will learn to better understand this situation (Myerson, 2010).

Mathematically...

Definition

The strategic form, or normal form, of a two-player game is given by a triplet (X, Y, A), where

- X is a nonempty set, the set of strategies of Player I
- Y is a nonempty set, the set of strategies of Player II
- A is a function $A : X \times Y \to \mathbb{R}^2$.

Definition

The normal form of an n-player game is given by $((X_i)_{i=1}^n, A)$ where

- X_i is a nonempty set, the set of strategies of Player $i \in N$.
- A is a function $A : \Pi_{i=1}^n X_i \to \mathbb{R}^n$.

Game: The Nash Equilibrium

Definition

- A play of the game where each strategy is a best reply to the other is a Nash equilibrium.
- A Nash equilibrium of a strategic game is a profile $a^* \in A$ of actions with the property that for every player $i \in N$ we have

$$u(a^*_{-i}, a^*_i) \geq u(a^*_i, a_i) \quad \forall\, a_i \in A_i$$

Nash Equilibria for the Games!

- The Prisoner's Dilemma: link
- The Battle of Sexes Game! link
- The Corruption Game! link
- The Wearing Mask Game! link
- The Akbar-Birbal Game! link

Few Remarks on Probability

- A and B are two events, then
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and
  $P(A \cap B) = P(A)P(B)$ and A and B are independent events.
- Expectation $E(X = x) = \sum_x xp(x)$.

Example

- X can invest in Share A with probability of 0.8 to have Rs. 500 in return.
- X can invest in Share B with probability of 0.2 to have Rs. 1500 in return.
- X can invest in Bond that gives him Rs. 300 in return.
- What will he do?

$$E(\text{Share}) = 0.8 \times 500 + 0.2 \times 1500 = 340.$$

Play safe? or Roll the Dice?

Nash Theorem

- Not all strategic games have Nash in pure strategy.
- Even some have Nash they are not unique, viz., Battle of Sexes.
- Existence of Nash Equilibrium proved using the Kakutani Fixed point Theorem.

Theorem

Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to \mathscr{P}(X)$ be a set-valued function for which,

- *for all $x \in X$ the set $f(x)$ is nonempty and convex;*
- *the graph of f is closed (i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$, and $y_n \to y$, we have $y \in f(x)$).*

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

Nash Theorem

Theorem

The strategic game has a Nash equilibrium if for all $i \in N$ the set A_i of actions of player i is a nonempty compact convex subset of a Euclidian space and the preference relation is continuous and quasi-concave on A_i.

Theorem

Every finite strategic game has a mixed strategy Nash equilibrium.

Mixed Nash for the Battle of Sexes

Husband

Wife: A	Movie (q)	Cricket $(1-q)$	Wife's Expected Payoff
Movie (p)	$(3,\ 1)$	$(0,\ 0)$	$3q + (1-q).0$
Cricket $(1-p)$	$(0,\ 0)$	$(1,\ 3)$	$0.q + (1-q).1$
Husband's Expected Payoff	$p.1 + (1-p).0$	$0.p + (1-p).3$	

Table: The Battle of Sexes

Now,

$$(1)\quad 2 \times q + 0 \times (1-q) = 0 \times q + 1 \times (1-q)$$

$$\Rightarrow q = \frac{1}{4} \rightarrow \quad \text{Mixed Strategy of Husband: } (1/4, 3/4).$$

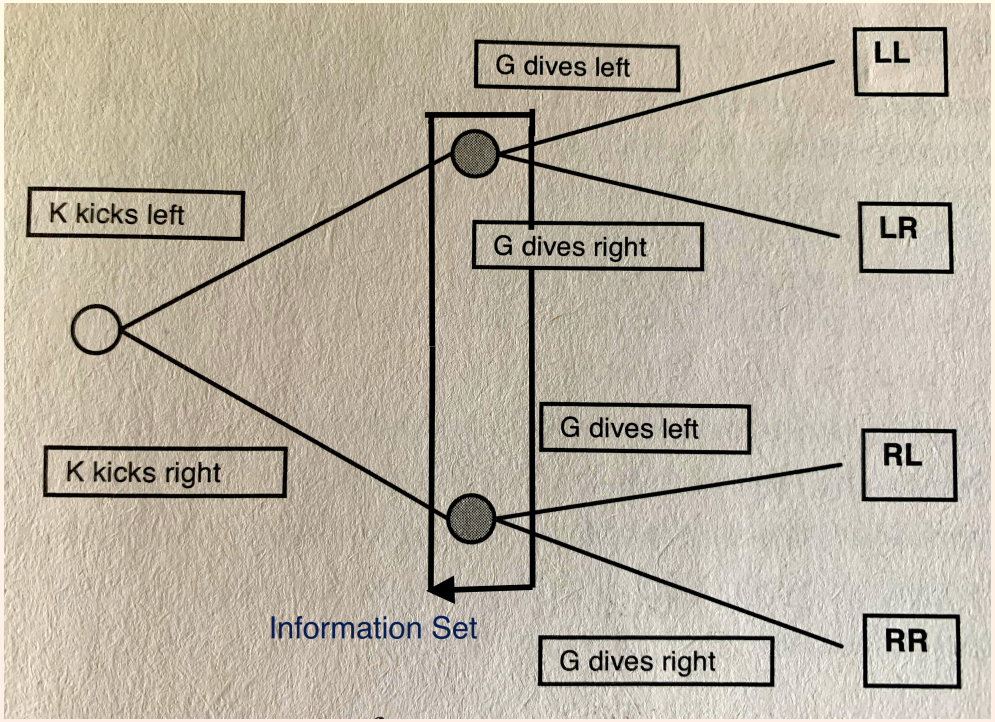
$$(2)\quad 1 \times p + 0 \times (1-p) = 0 \times p + 3 \times (1-p)$$

$$\Rightarrow p = \frac{3}{4} \rightarrow \quad \text{Mixed Strategy of Wife: } (3/4, 1/4).$$

Extensive Form Games

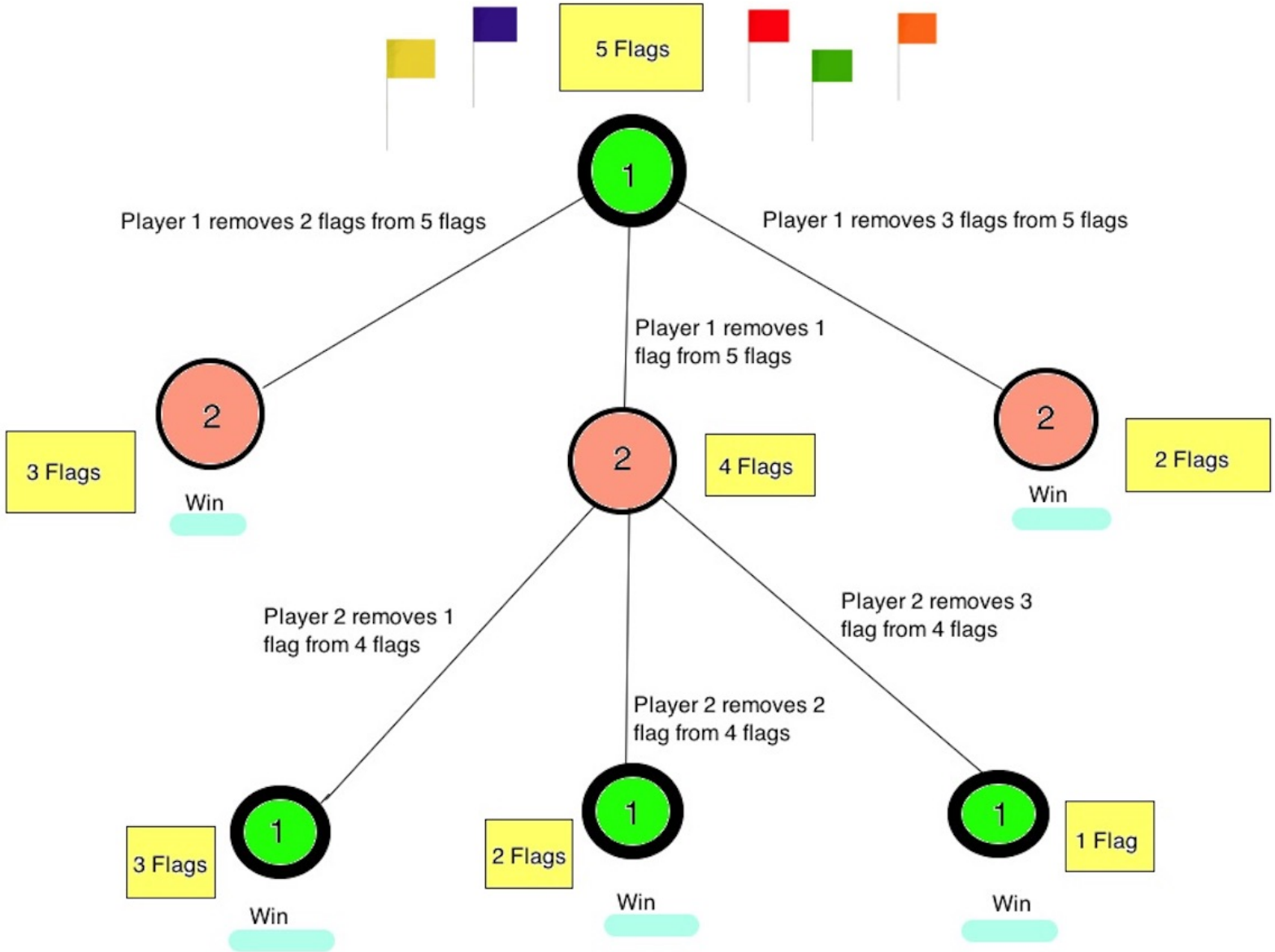
- Some situations require the movements of the play!
- Its not about outcomes and the strategies; The sequence is important!
- Usually represented by a Game Tree!!!

The Penalty-Kick Example



Information set has 2 nodes: Imperfect Information!

The Survival Thailand Example: Game with Perfect Information!



Cooperative Games

- Players make binding agreements to maximize individual utilities!
- Coalitions are formed, their worths generated.
- Total profits or costs should be shared among the players.
- Sometimes there are options to redistribute in the best possible way!(TU-Games)
- Sometimes there is only one way! (NTU Games)

Some History

- 0-500 AD : Babylonian Talmud—Compilation of Ancient law and Traditions.
- 1730 AD : Francis Waldegrave –Two Person Game and its Minimax Principle.
- 1838 AD : Augustine Cournot's "Researches in to the Mathematical Principles of Wealth" , Chapter 7; Dupoly and Restricted version of Nash Equilibrium.
- 1871 AD : Darwin in "The Descent of Man and Selection in Relation to Sex" –Evolutionary Biology.
- 1881 AD : Edgeworth's Mathematical Psychics...Contact curves $\rightrightarrows$ Core.
- 1913 AD : Zermelo –In Chess either a white or a black force a win or make a draw.
- 1921 AD : Emile Borel –4 notes on strategic games–maximin principle for mixed strategy.

Some History

- 1928 AD : Newmann proved the minimax...used Topology and Functional Calculus.
- 1930 AD : F. Zenthen (Book) : Problems of Monopoly and Economic Warfare–Nash Bargaining.
- 1944 AD : John von Newmann and Oscar Morgenstein –Theory of Games and Economic Behaviour. Cooperative Games introduced.
- 1950 AD : Prisoner's Dilemma –A. Tucker
- 1950 - 1953 AD : John Nash – 4 papers —Seminal contribution Non-cooperative games and Bargaining.
- 1952-53 AD : The Core –Shapley and Gillies (1953).

Some History

- 1954-55 AD : Differential Games by Rufu Isaac.
- 1959 AD : R. J. Aumann –Strong Equilibrium–1960 AD : Repeated Games.
- 1965 AD : R. Selten –Subgame Perfect Equilibrium
- 1967 AD : John Harsanyi –Bayesian Games
- 1972 AD : Maynard Smith – Evolutionary Games.
- 1994 AD : Selten, Harsanyi and Nash –Nobel for Economics.
- 1996 AD : Network Games - Jackson and Wolinsky, Goel.
- 2005 AD : Nobel to Eric Maskin –Mechanism Design.
- Last Decade : Network Games, Coalitional Hierarchies, Axiomatizations–

Thank You