MATHEMATICAL OLYMPIAD 2011 CATEGORY IV

Assam Academy of Mathematics
04 September 2011
All questions carry equal marks.

1.
- *Prove that for* $b \geq 4$, n, $n+2$ *and* $n+4$ *cannot all be primes.*
- *Find all primes* p *such that both* p *and* p^2+8 *are primes.*

2. *Find all positive integers* a *and* b *such that each of the equations* $x^2 - ax + b = 0$ *and* $x^2 - bx + a = 0$ *has distinct positive integral roots.*

3.
- *If* $x,\ y,\ z > 0$ *and* $x+y+z = 1$, *then show that* $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9$.
- *If in a triangle* ABC,
$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} = S = \frac{1}{2}(a+b+c).$$
Prove that triangle ABC *is equilateral.*

4.
- *If* $x = 2$ *is a root of* $84x^3 - 157x^2 - kx + 78 = 0$. *Find the value of* k *and the other roots.*
- *Find real numbers* a *and* b *if* $x^2 + x + 1$ *is a factor of* $2x^6 - x^5 + ax^4 + x^3 + bx^2 - 4x - 3$.

5. *Prove that* $f(n) = 1 - n$ *is the only integer valued function defined on integers such that*
- $f(f(n)) = n$ *for all* $n \in Z$ *and,*
- $f(f(n+2)+2) = n$ *for all* $n \in Z$ *and,*
- $f(0) = 1$.

6. *Construct* ΔABC, *given* $\angle A$, *side* AC *and the radius* r *of the inscribed circle. Justify your construction.*

7. ΔABC *is right angled at* C. *The internal bisectors of* $\angle A$ *and* $\angle B$ *meet* BC *and* CA *at* P *and* Q *respectively.* M *and* N *are the feet of the perpendiculars from* P *and* Q *to* AB. *Find* $\angle MCN$.

8.
- *Show that the total number of subsets of a set* S *with* n *elements is* 2^n.
- *Let* $A = \{a_1, a_2, a_3, \ldots, a_n\}$ *and* $B = \{b_1, b_2\}$. *Find the number of onto functions that can be defined from* A *to* B.

9. *Find the remainder when* 2^{1990} *is divided by* 1990.

10. *Find the triples* (a, b, c) *of positive integers such that*
$$(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c}) = 3.$$