An Introduction to Category theory

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Abstract

In this short expository article our aim is to make students familiar with the concepts of categories, functors and natural transformations.

1 Axioms for Categories

Without using any set theoretic approach and only using axioms we are going to describe categories viz *metacategories*. For that we first require to define *metagraph*.

Definition 1.0.1. A *metagraph* consists of objects say $a, b, c, \dots$ and arrows $f, g, h, \dots$ along with following two operations:
Domain, which assigns to each arrow f an object a, say $a = \operatorname{dom} f$
Codomain, which assigns to each arrow f an object b, say $b = \operatorname{codom} f$

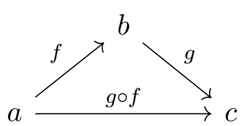
We can display these operations by means of an arrow diagram as follows

$$a \xrightarrow{f} b$$

Now we will define a *metacategory*.

Definition 1.0.2. A *metacategory* is a metagraph with additional two operations viz.
Identity, which assigns to each object a an arrow $Id_a = 1_a : a \to a$
Composition, which assigns to each pair $\langle g, f \rangle$ of arrows provided $\operatorname{dom} g = \operatorname{codom} f$, an arrow $g \circ f : \operatorname{dom} f \to \operatorname{codom} g$.

We can represent this operation by means of a diagram,



These operations in a *metacategory* are subject to the following axioms:
Associativity, for given objects and arrows in the following configuration

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

one always has the equality

$$h \circ (g \circ f) = (h \circ g) \circ f \tag{1}$$

This axiom assures that the *associative law* holds for the operation of composition whenever it makes sense.
Unit law, for all arrows $f : a \to b$ & $g : b \to c$ composition with the identity arrow gives

$$1_b \circ f = f \qquad \& \qquad g \circ 1_b = g \tag{2}$$

This axiom assures that the identity arrow 1_b of each object b acts as an identity for the operation of composition whenever it makes sense.

Example 1.0.1. Let us give an example of *metacategory.* We take *metacategory of sets* where *objects* are all sets and *arrows* are the function defined on them(set functions) along with identity function and composition of functions(Usual composition). For example a function $f : X \to Y$ consists of X as domain, Y as codomain and with a suitable set of ordered pairs $\langle x, fx \rangle$ which assigns for each $x \in X$,an element $fx \in Y$.
Similarly we can define *metacategory of all groups* where objects are all groups $G, H, K, \ldots$ and arrows are those functions $f : G \to H$ which are homomorphism of groups.

2 Categories

Now a category will mean any interpretation of category axioms(Associativity & Unital law) within Set theory thus making it distinguished from *metacategories.*

Definition 2.0.3. A *category* $\mathcal{C}$ consists of three ingredients: a class obj($\mathcal{C}$) of *objects* ,a set of *morphisms* $Hom(A, B)$ for every ordered pair (A, B) of objects and *composition* $Hom(A, B) \times Hom(B, C) \to Hom(A, C)$ denoted by $(f, g) \mapsto g \circ f = gf$, for every ordered triple A, B, C of objects.

These ingredients are subject to following axioms:

- the Hom sets are pairwise disjoint i.e. each $f \in Hom(A, B)$ has a unique *Domain* A & a unique *Codomain* B.
- for each object A,there exists an *identity morphism* $1_A \in Hom(A, A)$ such that $f \circ 1_A = f1_A = f$ and $1_B \circ f = 1_B f = f$;for each $f : A \to B$
- composition is associative i.e. for given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$,we have $h(gf) = (hg)f$.

Example 2.0.2. The category $\mathcal{C}(\mathcal{X})$ of partially ordered set $\mathcal{X}$,where obj $\mathcal{C}(\mathcal{X}) = \mathcal{X}$, $Hom_{\mathcal{C}(\mathcal{X})}(i, j)$ consists of a unique element say l^i_j if $i \leq j$ and is $\emptyset$,otherwise. Composition is given by $l^j_k l^i_j = l^i_k$ if $i \leq j \leq k$,because of reflexivity we have $1_i = l^i_i$ and composition makes sense because $\leq$ is transitive.

Example 2.0.3. Monoid is a category of one object. Thus each monoids is determined by the set of all it's arrows,by the identity arrow and by the rule of composition of arrows.Since any two arrows have a composite so monoids may be described as a set X with a binary operation $X \times X \to X$ which is associative and has an identity.

Example 2.0.4. A group is a category of one object and morphisms are homomorphisms and every morphism has a (two-sided) inverse under composition.

Example 2.0.5. **Grp**, is the category where objects are all groups and morphisms are homomorphisms between two groups.

Example 2.0.6. **Ab**, category of all abelian groups where objects are abelian groups and morphisms are group homomorphism between two abelian groups. In general by **Ab**, we represent abelian category where objects may be abelian groups, abelian rings, abelian varieties etc.

Example 2.0.7. **Rng**,category of rings with object as all rings with morphism as homomorphism(preserving units) between two rings.

Example 2.0.8. **Top**,category of topological spaces where obj***(Top)*** is all topological spaces and morphism are continuous maps between them.

Example 2.0.9. **Toph**,where obj***(Toph)*** is all topological spaces and morphism
$Hom_{Toph}(X, Y) = \{$the set of homotopy classes of continuous mapping from X to $Y\}$.

Example 2.0.10. **Top$_*$**,in this category obj*(Top_*)* consists of all ordered pairs (X, x_0) where X is a non-empty topological spaces & $x_0 \in X$ & morphisms are continuous maps $f : (X, x_0) \to (Y, y_0)$ where $f : X \to Y$ with $f(x_0) = y_0$ i.e. base point preserving maps.

3 Functors

Definition 3.0.4. If $\mathcal{C}$ and $\mathcal{D}$ are two categories,then a **functor** $T : \mathcal{C} \to \mathcal{D}$ is a function such that-
- if $a \in \text{obj}(\mathcal{C})$,then $T(a) \in \text{obj}(\mathcal{D})$,
- if $f : a \to a'$ in $\mathcal{C}$,then $T(f) : T(a) \to T(a')$ in $\mathcal{D}$,
- if $a \xrightarrow{f} a' \xrightarrow{g} a''$ in $\mathcal{C}$,then $T(a) \xrightarrow{T(f)} T(a') \xrightarrow{T(g)} T(a'')$ in $\mathcal{D}$ and

$$T(gf) = T(g)T(f)$$

-$T(1_a) = 1_{T(a)}$ for each $a \in \text{obj}(\mathcal{C})$.
We call this functor a **covariant functor**.

Example 3.0.11. Power set functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$. The object function assigns to each set X the usual power set $\mathcal{P}X$,where elements are all subsets $S \subset X$; it's arrow function assigns to each $f : X \to Y$ the map $\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y$ which sends each $S \subset X$ to its image $fS \subset Y$. Since both $\mathcal{P}(1_X) = 1_{\mathcal{P}X}$ and $\mathcal{P}(gf) = \mathcal{P}(g)\mathcal{P}(f)$,this defines a functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$.

Example 3.0.12. If $\mathcal{C}$ is a category,then the **identity functor** $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined by $1_{\mathcal{C}}A = A$ for all objects A and $1_{\mathcal{C}}f = f$ for all morphisms f. It's the simplest example of a functor.

Example 3.0.13. If $\mathcal{C}$ is a category and $A \in \text{obj}(\mathcal{C})$,then the Hom **functor** $T_A : \mathcal{C} \to \mathbf{Sets}$,denoted by $\text{Hom}(A, -)$ is defined by

$$T_A(B) = \text{Hom}(A, B) \ \ for\ all \ \ B \in \text{obj}(\mathcal{C})$$

and if $f : B \to B'$ in $\mathcal{C}$,then $T_A(f) : \text{Hom}(A, B) \to \text{Hom}(A, B')$ is given by

$$T_A(f) : h \mapsto fh$$

We call $T_A(f) = \text{Hom}(A, f)$ the induced map and denoted it by f_*.

Example 3.0.14. Singular homology in a given dimension n(n is a natural number)assigns to each topological space X an abelian group $H_n(X)$,the n-th homology group of X and also to each continuous map $f : X \to Y$ of spaces a corresponding homomorphism $H_n(f) : H_n(X) \to H_n(Y)$ of groups thus making H_n a functor $\mathbf{Top} \to \mathbf{Ab}$.

Example 3.0.15. Continuing with the previous example homotopic maps $f, g : X \to Y$ yield the same homomorphism $H_n(X) \to H_n(Y)$,so we can also regard H_n as a functor $\mathbf{Toph} \to \mathbf{Grp}$,defined on homotopy category.

Example 3.0.16. For any commutative ring K,we can attach the general linear group of $n \times n$ non-singular matrices with entries from K viz $\text{GL}_n(K)$. For any ring homomorphism $f : K \to K'$ we can associate a homomorphism $\text{GL}_n(f) : \text{GL}_n(K) \to \text{GL}_n(K')$ of groups and thus defines the functor GL_n:$\mathbf{CRng} \to \mathbf{Grp}$,where $\mathbf{CRng}$ is the category of commutative rings.

Definition 3.0.5. A functor which simply forgets some or all of the structure of an algebraic object is known as **forgetful functor**. Let us give an example to clarify this,

Example 3.0.17. A forgetful functor F:**Grp** $\rightarrow$ **Set** assigns to each group G ,the set $F\,G$ of its elements(thus forgetting the multiplication and hence the group structure) and assigns to each morphism $f : G \rightarrow G'$ of groups the same function f,regarded just as a function between sets.

Definition 3.0.6. A functor $T : A \rightarrow B$ is **full** when to every pair a, a' of objects of A and to every arrow $g : Ta \rightarrow Ta'$ of B,there is an arrow $f : a \rightarrow a'$ of A with $g = T\,f$.
It is very clear that composite of two full functors is also a full functor.

Definition 3.0.7. A functor $T : \mathcal{A} \rightarrow \mathcal{B}$ is **faithful** if for all $a, b \in \text{obj}(\mathcal{A})$, the functions $\text{Hom}_{\mathcal{A}}(a, b) \rightarrow \text{Hom}_{\mathcal{B}}(Ta, Tb)$ given by $f \mapsto T\,f$ are injections. Again composite of two faithful functors is a faithful functor.

Example 3.0.18. The previous forgetful functor **Grp** $\rightarrow$ **Set** is a faithful functor.However its not a full functor.

Definition 3.0.8. If $\mathcal{C}$ and $\mathcal{D}$ are two categories,then a **contravariant functor** $T : \mathcal{C} \rightarrow \mathcal{D}$ is a function such that-
- if $a \in \text{obj}(\mathcal{C})$,then $T(a) \in \text{obj}(\mathcal{D})$,
- if $f : a \rightarrow a'$ in $\mathcal{C}$,then $T(f) : T(a') \rightarrow T(a)$ in $\mathcal{D}$,
- if $a \xrightarrow{f} a' \xrightarrow{g} a''$ in $\mathcal{C}$,then $T(a'') \xrightarrow{T(g)} T(a') \xrightarrow{T(f)} T(a)$ in $\mathcal{D}$ and

$$T(gf) = T(f)T(g)$$

-$T(1_a) = 1_{T(a)}$ for each $a \in \text{obj}(\mathcal{C})$.

Example 3.0.19. Let us give an example of a contravariant functor, If $\mathcal{C}$ is a category and $B \in \text{obj}(\mathcal{C})$,then the **contravariant** Hom **functor** $T^B : \mathcal{C} \rightarrow$ **Sets**,denoted by $\text{Hom}(-, B)$,defined as

$$T^B(C) = \text{Hom}(C, B) \quad for\ all \quad C \in \text{obj}(\mathcal{C})$$

and if $f : C \rightarrow C'$ in $\mathcal{C}$,then $T^B(f) : \text{Hom}(C', B) \rightarrow \text{Hom}(C, B)$ is given by

$$T^B(f) : h \mapsto hf$$

3.1 Understanding the link between covariant and contravariant functor

As we have defined earlier what is meant by a covariant and a contravariant functor, we haven given examples 3.0.13 and 3.0.19 to show it in details. However reader must verify those two examples (left as an exercise!). However these two type of functors are connected by concepts like *duality* and *opposite categories*. We refer [1] to readers to understand the concept of opposite category.

4 Natural Transformations

Definition 4.0.1. Given two functors $S,T : \mathcal{C} \to \mathcal{B}$,a **natural transformation** is a function $\tau : S \to T$ which assigns to each object c in $\mathcal{C}$,an arrow $\tau_c = \tau\ c : Sc \to Tc$ of $\mathcal{B}$ in such a way that every arrow $f : c \to c'$ in $\mathcal{C}$ yields a commutative diagram,

$$\begin{array}{ccc} S\ c & \xrightarrow{\tau\ c} & T\ c \\ \big\downarrow{\scriptstyle S\ f} & & \big\downarrow{\scriptstyle T\ f} \\ S\ c' & \xrightarrow{\tau\ c'} & T\ c' \end{array}$$

When it holds,we also say that $\tau_c : Sc \to Tc$ is *natural* in c. A natural transformation is also called a *morphism of functors*.

Example 4.0.1. Let $\det_K M$ be the determinant of the $n \times n$ matrix M with entries from the commutative ring K,while K^* denotes the group of units of K. Thus M is non-singular when $\det_K M$ is a unit and $\det_K$ is a morphism $\mathrm{GL}_n\, K \to K^*$ of groups.As the determinant is defined by the same formula for all rings K, each morphism $f : K \to K'$ of rings leads to a commutative diagram

$$\begin{array}{ccc} \mathrm{GL}_n\, K & \xrightarrow{\det_K} & K^* \\ \big\downarrow{\scriptstyle \mathrm{GL}_n\, f} & & \big\downarrow{\scriptstyle f^*} \\ \mathrm{GL}_n\, K' & \xrightarrow{\det_{K'}} & K'^* \end{array}$$

this states that the transformation $\det : \mathrm{GL}_n \to (\ )^*$ is natural between two functors **CRng** $\to$ **Grp**.

Example 4.0.2. For each group G the projection $p_G : G \to \frac{G}{[G,G]}$ to the factor commutator group defines a transformation p from the identity functor on **Grp** to the factor commutator functor **Grp** $\to$ **Ab** $\to$ **Grp**.Moreover,p is natural because each group homomorphism $f : G \to H$ defines the evident homomorphism f' for which we get the following commutative diagram,

$$\begin{array}{ccc} G & \xrightarrow{p_G} & \frac{G}{[G,G]} \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} \\ H & \xrightarrow{p_H} & \frac{H}{[H,H]} \end{array}$$

5 Reference

[1] **Contravariance and opposites**, pg-33, *Constructions of Categories*, Categories for Working Mathematicians, S.Maclane.

6 Suggested reading

Interested students can read more about category theory from the book *Categories for Working Mathematicians* by S.Maclane.
Also we direct students to read the book - *A Survey of Modern Algebra* by Birkhoff and Maclane to grow more interests in algebraic systems and their connections with category theory.