# PROBLEMS IN GEOMETRY

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**1.** *For any triangle $ABC$ with circumradius $R$, prove that*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Extended Law of Sines

**2.** *In any triangle $ABC$, prove that $(ABC) = \frac{abc}{4R}$, where $(ABC)$ represents the area of the traingle $ABC$.*

**3.** *If three Cevians $AX, BY, CZ$, one through each vector of triangle $ABC$, are concurrent, then prove that*

$$\frac{BX}{XC} . \frac{CY}{YA} . \frac{AZ}{ZB} = 1.$$

G. Ceva

**4.** *Prove the converse of the above result.*

G. Ceva

**5.** *Let $AX$ be a Cevian of length $p$, dividing $BC$ into segments $BX = m$ and $XC = n$, then prove that $a(p^2 + mn) = b^2 m + c^2 n$.*

Stewart

**6.** *Let $ABC$ be an isoceles triangle having $\mid AB \mid = \mid AC \mid$. If $AB$ is an interior Cevian that intersects the circle circumscribed about $ABC$ at $S$, then describe the geometric locus of the center of the circle circumscribed about the triangle $BST$ where $\{T\} = AS \cap BC$.*

**7.** *Denote $P$ to be the set of points of the plane. Let $\star : P \times P \to P$ be the following binary operation: $A \star B = C$, where $C$ is the unique point in the plane such that $ABC$ is an oriented equilateral triangle whose orientation is counter clockwise. Show that $\star$ is an nonassociative and noncommutative operation satisfying the following 'medial' property: $(A \star B) \star (C \star D) = (A \star C) \star (B \star D)$.*

American Mathematical Monthly

**8.** *Show that there exists at most three points on the unit circle with the distance between any two being greater than $\sqrt{2}$.*

American Mathematical Monthly

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**9.** *Prove that a circumscribed quadrilateral of area $S = \sqrt{abcd}$ is inscribable.*

Putnam

**10.** *Consider $n$ points lying on the unit sphere. Prove that the sum of the squares of the lengths of all segments determined by the $n$ points is less than $n^2$.*

Kvant

**11.** *The sum of the vectors $\overrightarrow{OA_1}, \dots, \overrightarrow{OA_n}$ is zero, and the sum of their lengths is $d$. Prove that the perimeter of the polygon $A_1A_2\dots A_n$ is greater than $\frac{4d}{n}$.*

Kvant

**12.** *In a triangle $ABC$, let the internal bisectors of angles $B$ and $C$ meet the opposite sides $AC$ and $AB$ at $E$ and $F$ respectively. Suppose that $BE = CF$, then show that $AB = AC$.*

Steiner-Lehmus

**13.** *Points $A, B, C, D, E$ lie on circle $\omega$ and point $P$ lies outside the circle. The given points are such that:*

i. *Lines $PB$ and $PD$ are tangent to $\omega$.*
ii. *$P, A, C$ are collinear.*
iii. *$\overline{DE} \parallel \overline{AC}$.*

*Prove that $\overline{BE}$ bisects $\overline{AC}$.*

USAJMO 2011, Zuming Feng

**14.** *Given that $A, B, C$ are noncollinear lattice points in the plane such that the distances $AB, AC, BC$ are integers. What is the smallest possible value of $AB$?*

Putnam, 2010

**15.** *Let $ABC$ be a triangle such that its altitude $\overline{AA'}$ by $A$ lies inside the triangle. Prove that there exists a point $P$ on the segment $AA'$ such that the Cevians $\overline{BB'}$ and $\overline{CC'}$ through $P$ have the property $AB' = AC'$.*

Tahani Fraiwan-Mowaffaq Hajja, Math. Mag., 84(3), June 2011

**16.** *Let $a, b, c$ be the lengths of the sides of a triangle. Prove that*

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

**17.** *Let $a, b, c$ be the lengths of the sides of a triangle. Prove the inequality*

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

**18.** *Let $a$, $b$, $c$ be the lengths of a triangle with area $S$. Show that*

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Weitzenböck

**19.** *For any triangle $ABC$ with sides $a$, $b$, $c$ and area $F$, the following inequality holds.*

$$2ab + 2bc + 2ca - (a^2 + b^2 + c^2) \geq 4\sqrt{3}F.$$

Hadwiger-Finsler

**20.** *Let $p, q, r$ be positive real numbers and let $a, b, c$ denote the sides of a triangle with area $F$. Then, we have*

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq 2\sqrt{3}F.$$

Tsintsifas

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