PIGEONHOLE PRINCIPLE

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1. *Prove that if $n+1$ objects are put into n boxes, then at least one box contains two or more of the objects.*

Simple form of Pigeonhole Principle

2. *Among 13 people prove that there will be two who have their birthdays in the same month.*

3. *There are n married couples. How many of the $2n$ people must be selected in order to guarantee that one has selected a married couple?*

4. *Let X and Y be finite sets and let $f: X \to Y$ be a function from X to Y.*

- *If X has more elements than Y, then f is not one to one.*
- *If X and Y have the same number of elements and f is onto, then f is one to one.*
- *If X and Y have the same number of elements and f is one to one, then f is onto.*

Function form of Pigeonhole Principle

5. *A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day, but in order not to tire himself, he decides to not play more than 12 games during any calender week. Show that there exists a succession of (consecutive) days during which he will have played exactly 21 games.*

6. *Show that there is a succession of days, during which the chess master will have played exactly k games, for each $k = 1, 2, \ldots, 21$. Is it possible to conclude that there is a succession of days during which the chess master will have played exactly 22 games?*

7. *From the integers $1, 2, \ldots, 200$, we choose 101 integers. Show that among the integers choosen, there are two such that one of them is divisble by the other.*

8. *If 100 integers are choosen from $1, 2, \ldots, 200$, and one of the integers choosen is less than 16, then there are two choosen numbers such that one of them is divisble by the other.*

9. *Show that if $n+1$ integers are choosen from the set $\{1, 2, \ldots, 2n\}$ then there are always two that differ by 1.*

10. *Show that if $n+1$ integers are choosen from the set $\{1, 2, \ldots, 2n\}$ then there are always two which differ by at most 2.*

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11. *Generalize the above two problems.*

12. *Prove that of any five points choosen within a square of side length* 2*, there are two whose distance apart is at most* $\sqrt{2}$*.*

13. *A collection of subsets of* $\{1, 2, \ldots, n\}$ *has the property that each pair of subsets has at least one element in common. Prove that there are at most* 2^{n-1} *subsets in the collection.*

14. *Let* m *and* n *be relatively prime positive integers, and let* a *and* b *be integers where* $0 \leq a \leq m-1$ *and* $0 \leq b \leq n-1$*. Then there is a positive integer* x *such that the remainder when* x *is divided by* m *is* a*, and the remainder when* x *is divided by* n *is* b*.*

Chinese Remainder Theorem

15. *A rational number* $\frac{a}{b}$ *has a decimal expression that eventually repeats.*

16. *Let* $q_1, q_2, \ldots, q_n$ *be positive integers. If* $q_1 + q_2 + \ldots + q_n - n + 1$ *objects are put in* n *boxes then the* i*-th box contains at least* q_i *objects.*

Strong form of Pigeonhole Principle

17. *If* $n(r-1)+1$ *objects are put into* n *boxes, then at least one of the boxes contains* r *or more of the objects.*

18. *If the average of* n *nonnegative integers* $m_1, m_2, \ldots, m_n$ *is greater than* $r-1$*, then at least one of the integers is greater than or equal to* r*.*

19. *If the average of* n *nonnegative integers* $m_1, m_2, \ldots, m_n$ *is less than* $r+1$*, then at least one of the integers is less than* $r+1$*.*

20. *A basket of fruits is being arranged out of apples, bananas and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least* 8 *apples or at least* 6 *bananas or at least* 9 *oranges?*

21. *If the average of* n *nonnegative integers* $m_1, m_2, \ldots, m_n$ *is at least equal to* r*, then at least one of the integers* m_i*, satisfies* $m_1 \geq r$*.*

22. *Two disks, one smaller than the other, are each divided into* 200 *congruent sectors. In the larger disk,* 100 *of the sectors are choosen arbitarily and painted red; the other* 100 *sectors are painted blue. In the smaller disk, each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The small disk is then placed on the larger disk so that their centers coincide. Show that it is possible to allign the two disks so that the number of sectors of the smaller disk whose colour matches the corresponding sector of the larger disk is at least* 100*.*

23. *Show that every sequence* $a_1, a_2, \ldots, a_{n^2+1}$ *of* n^2+1 *real numbers contains either an increasing or a decreasing subsequence of length* $n+1$*.*

Erdös-Szekeres

24. *Suppose that* n^2+1 *people are lined up shoulder to shoulder in a straight line. Then it is always possible to choose* $n+1$ *of the people to take one step forward so that going from left to right their heights are increasing (or decreasing).*

25. *Suppose that the mn people of a marching band are standing in a rectangular formation of m rows and n columns in such a way that in each row each person is taller than the one to his or her left. Suppose that the leader rearranges the people in each column in increasing order of height from front to back. Show that the rows are still arranged in increasing order of height from left to right.*

26. *Of six (or more) people, either there are three, each pair of whom are acquainted, or there are three, each pair of whom are unacquainted.*

Ramsey

References:

Richard A. Brualdi, *Introductory Combinatorics*, 4th ed., Pearson, 2004