Assorted Problems

Manjil P. Saikia
Department of Mathematical Sciences
Tezpur University, Napaam
Pin 784028, India
manjil@gonitsora.com

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1. (Mathematics Magazine(MM)) Find all values of n for which the sum $1+2+\cdots+n$ is an integer power of 10.

2. (MM) For which positive integer n do there exist positive integer solutions x, y to the diophantine equation $4xy - x + y = n$?

3. (American Mathematical Monthly (AMM)) Let p be an odd prime and $k \in \mathbb{Z}_+$. Show that there exists a perfect square the last k digits of whose expansion in base p are 1.

4. Any natural number greater than 6 can be written as a sum of two numbers that are relatively prime.

5. (IMO) Prove that it is impossible to extract an infinite AP from the sequence $S = \{1, 2^k, 3^k, \ldots, n^k, \ldots\}$ where $k \geq 2$.

6. Replace S by $S = \{a_{n+2} = pa_{n+1} + qa_n\}$ for $1 \leq p \leq q+1$ and solve.

7. (Putnam 1964) Let u_n be the least common multiple of the first n terms of a strictly increasing sequence of positive integer. Prove that
$$\sum_{n=1}^{\infty} \frac{1}{u_n} \leq 2.$$
Find a solution for which equality holds.

8. (Kvant) Let $\sigma(n)$ denote the sum of the divisors of n. Prove that there exists infinitely many integers n such that $\sigma(n) > 2n$ or even stronger $\sigma(n) > 3n$. Prove also that $\sigma(n) < n(1 + \log n)$.

9. (USA) Let $a_i \in \mathbb{N}$ and $\gcd(a_i, a_j) = 1$ and a_i's are not primes, show that
$$\sum_{i+1}^{n} \frac{1}{a_i} < 2.$$

10. (AMM) Find the gcd of $\binom{1}{2n}, \binom{3}{2n}, \binom{5}{2n}, \cdots, \binom{2n-1}{2n}$.

11. Find the minimum number of elements that must be deleted from the set $\{1, 2, \ldots, 2005\}$ such that the set of the remaining elements doesn't contain two elements together with their product. Does there exist, for any k, an AP with k terms in the infinite sequence
$$1, \frac{1}{2}, \ldots, \frac{1}{2005}, \ldots, \frac{1}{n}, \ldots?$$

12. Find an example of a sequence of natural numbers $1 \leq a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots$ with the property that every $m \in \mathbb{Z}_+$ can be uniquely written as $m = a_i - a_j$ for $i, j \in \mathbb{Z}_+$.

13. (AMM) Consider the set of $2n$ integers $\{\pm a_1, \pm a_2, \ldots, \pm a_n\}$ and $m < 2^n$. Show that we can choose a subset S such that

    (a) The two numbers $\pm a_i$ are not both in S;

    (b) The sum of all elements of S is divisible by m.

14. (AMM) For a finite graph G we denote by $Z(G)$ the minimal number of colours needed to colour all its vertices such that adjacent vertices have different colours. Prove that $Z(G) \geq \frac{p^2}{p^2-2q}$ holds if G has p vertices and q edges.

15. Write $[a_1, a_2, \ldots, a_n]$ in terms of various $\gcd(a_{i_1}, a_{i_2}, \ldots, a_{i_j})$ for subsets of $\{a_1, a_2, \ldots, a_n\}$.

16. (Romania) The set $M = \{1, 2, \ldots, 2n\}$ is partitioned into k sets $M_1, M_2, \ldots, M_k$ where $n \geq k^3 + k$. Show that there exists $i, j \in \{1, 2, \ldots, k\}$ for which we can find $k+1$ distinct even numbers $2r_1, 2r_2, \ldots, 2r_{k+1} \in M_i$ with the property that $2r_1 - 1, 2r_2 - 1, \ldots, 2r_{k+1} - 1 \in M_j$.

References

[1] V. Boju, L. Funar, *The Math Problems Notebook*, Brikhäuser, 2007.

[2] http://www.manjilsaikia.in/olympiads