Functional Equations

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1. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that
   (a) $f(2) = 2$;
   (b) $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$;
   (c) $f(m) < f(n)$ for $m < n$.
2. Replace (b) in the above by $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1$ and then solve for f.
3. Replace (a) in the above by $f(2) = 3$ and see what happens?
4. What happens if there is no (c) in the above?
5. (IMO 1997) If $f\colon \mathbb{N} \to \mathbb{N}$ is such that $f(n+1) > f(f(n))$ for all $n \in \mathbb{N}$, prove that $f(n) = n$ for all $n \in \mathbb{N}$.
6. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(f(m) + f(n)) = m + n$ for all $m, n \in \mathbb{N}$.
7. (IMO 1998) Consider all functions $f\colon \mathbb{N} \to \mathbb{N}$ satisfying $f(m^2 f(n)) = n(f(m))^2$ for all $m, n \in \mathbb{N}$. Determine the least possible value of $f(1998)$.
8. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that $f(m+n) + f(mn-1) = f(m)f(n) + 2$ for all $m, n \in \mathbb{Z}$.
9. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that $f(m+n) + f(mn) = f(m)f(n) + 1$ for all $m, n \in \mathbb{Z}$.
10. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that $f(m+n) + f(mn-1) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$.
11. Determine all functions $f\colon \mathbb{Z} \setminus \{0, 1\} \to \mathbb{R}$ which satisfy $f(x) + f(\frac{1}{1-x}) = \frac{2(1-2x)}{x(1-x)}$ valid for all $x \neq 0$ and $x \neq 1$.
12. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that
    (a) $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$,
    (b) $f(\frac{1}{x}) = \frac{f(x)}{x^2}$ for all $x \neq 0$.

    Prove that $f(x) = cx$ for all $x \in \mathbb{R}$ for some constant c.
13. (IMO 1992) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x^2 + f(y)) = f(x)^2 + y$ for all $x, y \in \mathbb{R}$.
14. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ which satisfy $f(xf(x) + f(y)) = f(x)^2 + y$ for all $x, y \in \mathbb{R}$.

15. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(f(x)+y) = f(x^2-y) + 4f(x)y$ for all $x, y \in \mathbb{R}$.

16. (Nordic Contest 1998) Find all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in \mathbb{Q}$.

17. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that

    (a) $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$,

    (b) f is monotonic in $\mathbb{R}$.

    Prove that f is linear, that is there exist a constact c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

18. Let $n \geq 2$ be a fixed integer. Determine all bounded functions $f\colon (0, a) \to \mathbb{R}$ which satisfy

$$f(x) = \frac{1}{n^2}\left\{ f\left(\frac{x}{n}\right) + f\left(\frac{x+a}{n}\right) + \cdots + f\left(\frac{x+(n-1)a}{n}\right) \right\}.$$

19. Find all strictly monotone functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(f(x)+y) = f(x+y) + f(0)$$

    for all $x, y \in \mathbb{R}$.

20. Find all continous functions $f\colon \mathbb{R} \to \mathbb{R}_0$ such that $f(x+y) = f(x) + f(y) + f(x)f(y)$ for all $x, y \in \mathbb{R}$.

References

[1] B. J. Venkatachala, *Functional Equations: A Problem Solving Approach*, Prism Books Pvt. Ltd., 2002

[2] http://www.manjilsaikia.in/olympiads