Geometry

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1. (Menelaus) If a transversal cuts the sides BC, CA, AB (suitably extended) of $\triangle ABC$ in points D, E, F, respectively, then prove that $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$.

2. (Converse of Menelaus) If points D, E, F are taken on the sides of $\triangle ABC$ such that the above equation holds, then prove that D, E, F are collinear.

3. (Stewart) In $\triangle ABC$, D is the midpoint of BC, then prove that $AB^2 + AC^2 = 2AD^2 + 2DC^2$.

4. (British 1995) $\triangle ABC$ has a right angle at C. The internal angle bisectors of $\angle BAC$ and $\angle ABC$ meet BC and CA at point P and Q respectively. The points M and N are feet of the perpendiculars from P and Q to AB. Find $\angle MCN$.

5. (Leningrad 1988) Squares $ABDE$ and $BCFG$ are drawn of outside $\triangle ABC$. Prove that $\triangle ABC$ is isoceles if $DG \parallel AC$.

6. The lengths of the sides of a quadrilateral are positive integers. The length of each side divides the sum of the other three lengths. Prove that two of the sides have the same length.

7. (IMO 1985) A circle has centre on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

8. Quadrilaterals $ABCP$ and $A'B'C'P'$ are inscribed in two concentric circles. If $\triangle ABC$ and $\triangle A'B'C'$ are equilateral, then prove that $P'A^2 + P'B^2 + P'C^2 = PA'^2 + PB'^2 + PC'^2$.

9. Let the incribed circle of $\triangle ABC$ touches side BC at D, side CA to E and side AB at F. Let G be the foot of the perpendicular from D to EF. Show that $\frac{FG}{EG} = \frac{BF}{CE}$.

10. (Putnam 1996) Let C_1 and C_2 be circles whose centers are 10 units apart and whoose radii are 1 and 3. Find the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the mod point of the line segment XY.

11. (IMO 1982) Diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N respectively, so that $\frac{AM}{AC} = \frac{CN}{CE} = r$. Determine r, if b, M and N are collinear.

12. In $\triangle ABC$, suppose $AB > AC$. Let P and Q be the feet of the perpendiculars from B and C to the angle bisectors of $\angle BAC$ respectively. Let D be on line BC such that $DA \perp AP$. Prove that lines BQ, PQ and AD are concurrent.

13. (IMO 1995) Let A, B, C and D be four distinct points on a line, in that order. The circles with diameter AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M and the line BP intersects the circle with diameter BD at the points M and N. Prove that the lines AM, DN and XY are concurrent.

14. (Chinese Team Selection Test 1996) The semicircle with BC of $\triangle ABC$ as diameter intersects sides AB and AC at points D and E respectively. Let F and G be the feet of the perpendicular from D, E to side BC respectively. Let M be the intersection of DG and EF. Prove that $AM \perp BC$.

15. (IMO 1985) A circle with center O passes through the vertices A and C of $\triangle ABC$ and intersects the segments AB and AC again at distinct points K and N respectively. The circumradius of $\triangle ABC$ and $\triangle KBN$ intersect at exactly two distinct points M and N. Prove that $OM \perp MB$.

References

[1] K. Y. Li, *Math Problem Book I*, Hong Kong Mathematical Society: IMO(HK) Committee, 2001

[2] http://www.manjilsaikia.in/olympiads