Let's Count!

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1. If we have two finite sets A and B, and if f is a bijective mapping from A to B, then show that the sets A and B have the same cardinality.

2. In a simple graph $G(E, V)$ on n vertices, show that
$$\sum_{u \in V} \deg v = 2 \mid E \mid .$$

3. In a simple graph on n vertices, show that there must exist at least two vertices with the same degree.

4. (Ramsey Theorem) Show that in a gathering of six people, there will always be three poeple who either know each or do not know each other. What happens if there are only five people?

5. If $G(V, E)$ is a graph in which each vertex has degree at least 2, then show that it contains a cycle.

6. In a symmetric simple random walk, what are the total number of paths from $(0, 0)$ to (n, r), where $n, r \in \mathbb{N}$ and $r \leq n$?

7. In a symmetric simple random walk, show that the number of paths from $(0, 0)$ to (n, r) which touch or cross the x-axis is same as the number of paths from $(1, -1)$ to (n, r).

8. (Ballot Box Problem) Suppose that in a ballot a candidate P scores p votes and a candidate Q scores q votes, where $p > q$. Then show that the probability that throughout the counting there are always more votes for P than Q is $\frac{p-q}{p+q}$.

9. Show that $\mathbb{Q}$ is countable. Hence, or otherwise show that the set $\mathbb{Z} \times \mathbb{Z}$ is also countable.

10. How many integral solutions are possible for the equation $2x + 3y = 2014$, where $x, y \in \mathbb{N}$.

11. Suppose we toss a coin, and continue tossing it until head occurs and then we stop. What is the probability that we eventually stop?

12. Let us consider the first 100 natural numbers arranged serially in a line. Then we put a permutation of those 100 natural numbers serially just below the first line. We say that a match occurs at the i-th position if the i-th number in both the lines are identical. What is the probability that atleast one match occurs?

13. Without using any algebraic techniques, prove that $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$ for all positive integers n.

14. (Bay Area Math Meet) How many subsets of the set $\{1, 2, 3, \ldots, 30\}$ have the property that the sum of their elements is greater than 232?

15. (American Mathematical Monthly) Set $\pi(n)$ for the number of primes less than or equal to n. Prove that there are at most $\pi(n)$ numebrs $1 < a_1 < a_2 < \cdots < a_k \leq n$ with $\gcd(a_i, a_j) = 1$.

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