Polynomials

Manjil P. Saikia
Department of Mathematical Sciences
Tezpur University, Napaam
Pin 784028, India
manjil@gonitsora.com

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1. (Division Algorithm) If $f(x)$ be a polynomial of degree n, then there exists polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)q(x) + r(x)$ for some polynomial $g(x)$ such that $deg\ r < deg\ g$ or $r(x) = 0$.

2. (Rational Roots Theorem) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ have integer co-efficients and let $z \in Z$, then $f(z) = 0$ if and only if $z \mid a_0$. Further if, $a_n = 1$, then each rational root of f is an integer.

3. (Viete) For any polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, the sum of the roots taken k at a time is $(-1)^k \frac{a_{n-k}}{a_n}$.

4. Find all polynomials $P(x)$ such that $xP(x-1) = (x-15)P(x)$.

5. Find a closed form expression for $F(z) = \sum_{n=0}^{\infty} f_n z^n$, where f_n's are Fibonacci numbers.

6. Find all polynomials $P(x, y)$ in two variables which satisfy $P(x, y) = P(x+1, y+1)$.

7. Solve $x(3y-5) = y^2 + 1$ in integers.

8. If $x, y, z \in \mathbb{Q}$ for which $x^3 + 3y^3 + 9z^3 - 9xyz = 0$, prove that $x = y = z = 0$.

9. Let the polynomial $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + 1$ have non-negative coefficients and n real zeroes. Prove that $f(2) \geq 3^n$.

10. $p(x)$ and $q(x)$ are polynomials which satisfy the identity $p(q(x)) = q(p(x))$ for all x. If the equation $p(x) = q(x)$ has no real solutions show that the equation $p(p(x)) = q(q(x))$ also has no real solutions.

11. Let $p(x)$ be a polynomial over $\mathbb{R}$ of even degree n for which $p(x) \geq 0$ for all x, prove that

$$p(x) + p'(x) + \cdots + p^{(n)}(x) \geq 0$$

for all x.

12. Is there a set of real numbers u, v, w, x, y, z such that

$$u^2 + v^2 + w^2 + 3(x^2 + y^2 + z^2) = 6$$

and

$$ux + vy + wz = 2?$$

13. Show that, if p is an odd prime and k is a positive integer, then

$$x^p + 1 \mid (z^{p-1} - 1)(z^{p-2} - z^{p-3} + \cdots + z - 1)^k + (z+1)z^{(p-1)k-1}.$$

14. Observe that $8^3 - 7^3 = (2^2 + 3^2)^2$ and $105^3 - 104^3 = (9^2 + 10^2)^2$. Show that, if the difference of two consecutive cubes is a square, then it is the square of the sum of two successive squares.

15. Show that $2[(x-y)(x-z) + (y-z)(y-x) + (z-x)(x-y)]$ can be expressed as the sum of three squares.

16. If $x, y, z > 0$, show that
$$\frac{1}{x(1+y)} + \frac{1}{y(1+z)} + \frac{1}{z(1+x)} \geq \frac{3}{1+xyz}.$$

17. Reduce to lowest terms
$$\frac{(ab-x^2)^2 + (ax+bx-2x^2)(ax+bx-2ab)}{(ab+x^2)^2 - x^2(a+b)^2}.$$

18. Solve $x^5 + y^5 = 33, x + y = 3$.

References

[1] E. J. Barbeau, *Polynomials*, Springer, 1989.

[2] http://www.manjilsaikia.in/olympiads