

PROBLEMS IN ELEMENTARY NUMBER THEORY–CATEGORY III

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1. Let a_1, a_2, \dots, a_n be non-negative integers, then $(a_1 + a_2 + \dots + a_n)!$ is divisible by $a_1! \cdot a_2! \cdot \dots \cdot a_n!$.

Hojoo Lee, Tom Lovering and Cosmin Pohoată

2. Let m and n be arbitrary non-negative integers. Prove that $\frac{(2m)!(2n)!}{m!n!(m+n)!}$ is an integer.

IMO, 1972

3. There are n windows in a classroom which are all closed and numbered 1 through n . The teacher calls n students and then asks the n -th student to either close an open window or open a closed one whose number is a multiple of n . After the whole exercise which windows will be left open?

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4. Does there exist an infinite set A of positive integers with the property that the sum of the elements of any finite subset of A is not a perfect power?

Kvant, 1982

5. The $2n$ move chess game has the same rules as the regular one, with only one exception, each player has to make $2n$ consecutive moves at a time. Prove that white (who plays first) always has a non-losing strategy.

Manjil P. Saikia, Bangladesh Mathematical Olympiad Training Camp 2008

6. Prove that $\forall n \in \mathbb{Z}^+$,

$$\lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{n+2}{2^2} \rfloor + \dots + \lfloor \frac{n+2^k}{2^k+1} \rfloor + \dots = n.$$

IMO, 1968

7. If x is a positive real number and n is a positive integer, prove that,

$$\lfloor nx \rfloor \geq \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \dots + \frac{\lfloor nx \rfloor}{n}.$$

USAMO, 1981

8. Find the sum $1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n!$.

Canada, 1969

9. If $n > 1$ then prove that $n^4 + 4^n$ is never a prime.

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Kürschak, 1978

10. Show that the equation $x^2 + y^2 + z^2 = 2xyz$ has no integral solutions except $x = y = z = 0$.

Arthur Engel

11. If a, b and $q = \frac{a^2+b^2}{ab+1}$ are integers, then prove that q is a perfect square.

IMO, 1988

12. Solve the system of equations:

$$x + \lfloor y \rfloor + \{z\} = 200.0$$

$$\{x\} + y + \lfloor z \rfloor = 190.1$$

$$\lfloor x \rfloor + \{y\} + z = 178.8.$$

Australia, 1999

13. If $a \equiv b \pmod{n}$, then show that $a^n \equiv b^n \pmod{n^2}$.

Titu Andreescu, Dorin Andrica and Zuming Feng

14. Find the number of positive integers x which satisfy the equation:

$$\lfloor \frac{x}{99} \rfloor = \lfloor \frac{x}{101} \rfloor.$$

RMO, 2001

15. Let a, b and c be three natural numbers such that $a < b < c$ and $\gcd(c-a, c-b) = 1$. Suppose there exists an integer d such that $a + d, b + d$ and $c + d$ form the sides of a right angled triangle. Prove that there exists integers l and m such that $c + d = l^2 + m^2$.

RMO, 2007

16. Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every positive integer n .

IMO, 1959

17. If the distinct prime factors of n are p_1, p_2, \dots, p_k , then prove that

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right).$$

Leonhard Euler

18. Prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}.$$

Catalan

19. Let m and n be positive integers such that

$$\frac{m}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that $1979 \mid m$.

IMO, 1979

20. Let p be a prime number and m and n be two integers considered in the following way, $m = a_k p^k + a_{k-1} p^{k-1} + \dots + a_1 p + a_0$ and $n = b_l p^l + b_{l-1} p^{l-1} + \dots + b_1 p + b_0$, where all a_i, b_i 's are non-negative integers less than p . Then prove that,

$$\binom{m}{n} = \prod_{i=0}^{\max(k,l)} \binom{a_i}{b_i}.$$

E. Lucas

21. If p is a prime, then prove that for any integer a , $p \mid a^p + (p-1)!a$ and $p \mid (p-1)!a^p + a$.

David M. Burton

22. If p is an odd prime, then prove that

$$1^p + 2^p + \dots + (p-1)^p \equiv 0 \pmod{p}.$$

David M. Burton

23. Find the last two digits of 27^{1729} .

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24. Show that there are infinitely many non-zero solutions to the equation $a^3 + b^3 + c^3 + d^3 = 0$.

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25. If A and B are two Facebook friends then AB forms a Facebook line. $ABCD$ will be a Facebook polygon if A and B , B and C , C and D and A and D are friends. We can draw diagonals joining two points iff they are Facebook friends. Find out the largest sample size of people needed to form a Facebook quadrilateral with no diagonals.

Mohaimin Elements and Manjil P. Saikia

26. Prove that a natural number $p > 1$ is a prime iff $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$ is divisible by p for every non-negative integer n .

Manjil P. Saikia and Jure Vogrinc

27. Prove that a natural number $p > 1$ is a prime iff $\binom{q}{p} - \lfloor \frac{q}{p} \rfloor$ is divisible by p for every prime number q .

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28. Prove that if given any natural number n , the set $\{1, 2, 3, \dots, 2^{n+1}\}$ can be partitioned into two subsets, A_n and B_n , each of size 2^n , such that $\sum_{a \in A_n} a^k = \sum_{b \in B_n} b^k$ for $k = 1, 2, \dots, n$.

B. Sury

29. Show that $7x + 9y = n$ has non-negative integer solutions $\forall n > 47, n \in \mathbb{N}$.

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30. Prove that the number of representations of $8n+4$ as a sum of four odd squares is twice that when it is represented as the sum of four even squares.

Nayandeep Deka Baruah, Shawn Cooper and Micheal D. Hirshchorn