Geometry - An Olympiad Approach

Manjil P. Saikia
Department of Mathematical Sciences
Tezpur University
manjil@gonitsora.com

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1   Introduction

Geometry is one of the oldest and most essential subjects in all of mathematics. Unfortunately, nowadays the subject ceases to exist beyond the high school textbooks. Olympiads provides a much needed break from the traditional geometry problems as we shall see various beautiful results and their applications in the sections to follow. Geometry as a systematic study was first brought forth by the efforts of the Greek mathematician **Euclid** in around 300 B.C. when he first studied it as a form of the deductive method. Euclid in his most famous books **Elements** devoted much to the structure and formation of what is now called **Euclidean geometry**. The basis for his results were a set of axioms and propositions from which most of high school geometry follows quite easily. We state below the axioms of Euclid for completeness and also because of their inherent beauty.

1. Through any two points is a unique line.

2. A finite part of a line may be indefinitely extended.

3. There exists circles with any centre and radius.

4. All right angle triangles are equal to one another.

5. Let l be a line and P be a point not on l. Then all lines through P meet l, except just one, the parallel.

The last of the above axioms is also called **Playfair's axiom** and is a topic of much debate in mathematics. There has been new types of geometries defined on the basis of this fifth axiom, which we shall not go into details.

It should be mentioned that various other mathematicians has gives different sets of axioms and tried to form the geometry as we know it. For a good readable account of the set of axioms given by the great German mathematician **David Hilbert**, the reader can look into [1].

2 Basic Theorems

In this section we shall state without proof a few important theorems which are quite useful for solving mathematical olympiad problems. For the proofs the reader can look into [3].

Theorem 2.1 (Ceva)**.** *If points D, E, F are taken on the sides BC, CA, AB of $\triangle ABC$ so that the lines AD, BE, CF are concurrent at a point O, then*

$$\frac{BD.CE.AF}{DC.EA.FB} = 1. \tag{1}$$

Theorem 2.2 (Converse of Ceva's Theorem)**.** *If three points D, E, F taken on the sides BC, CA, AB of a $\triangle ABC$ are such that (1) is satisfied, then AD, BE, CF are concurrent.*

Theorem 2.3 (Menelaus)**.** *If a transversal cuts the sides BC, CA, AB (suitably extended) of $\triangle ABC$ in points D, E, F, respectively, then*

$$\frac{BD}{DC}.\frac{CE}{EA}.\frac{AF}{FB} = -1. \tag{2}$$

Theorem 2.4 (Converse of Menelaus' Theorem)**.** *If points D, E, F are taken on the sides of $\triangle ABC$ such that (2) holds, then D, E, F are collinear.*

Theorem 2.5 (Pythagoras)**.** *In any right angled triangle, the square on the hypotenuse is equal to the sukm of the squares on the other sides.*

Theorem 2.6 (Stewart)**.** *If point D divides the base BC of $\triangle ABC$ in the ratio $\frac{BD}{DC} = \frac{n}{m}$, then*

$$m.AB^2 + n.AC^2 = m.BD^2 + n.CD^2 + (m+n).AD^2.$$

Theorem 2.7 (Apollonius)**.** *In $\triangle ABC$, D is the midpoint of BC, then*

$$AB^2 + AC^2 = 2AD^2 + 2DC^2.$$

Theorem 2.8 (Euler). *In any $\triangle ABC$, $OI^2 = R^2 - 2Rr$, where O, I are the centers and R, r are the radii, respectively of the circumcircle and incircle of $\triangle ABC$.*

Theorem 2.9 (Ptolemy). *The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by the pairs of its opposite sides.*

Theorem 2.10 (Extension of Ptolemy's Theorem). *If $ABCD$ is a quadrilateral which is not cyclic, then $BC.AD + AB.CD > AC.BD$.*

Theorem 2.11 (Brahmagupta). *If in $\triangle ABC$, AD is the altitude and AE is the diameter of the circumcircle through A, then*

$$AB.AC = AD.AE.$$

Theorem 2.12 (Steiner-Lehmus). *If in a triangle two internal angle bisectors are equal, then the triangle is isoceles.*

3 Problems

Now that we have learnt some amount of basic geometry and discussed the above problems, the reader should be in a position to solve the following problems. It is important to remember that a good mathematical problem is one which attacks you back while you try to solve it, so perseverance and a keen eye for details are the hallmark of a good problem solver. Mathematical Olympiad problems, specially the geometrical ones are difficult nuts to crack, the selection below is a varied one ranging from quite easy ones to diabolically difficult ones. Best of luck!

1. Prove all the theorems in the previous section.

2. Prove that the medians, internal angle bisectors, external angle bisectors and altitudes of a triangle are concurrent.

3. The incircle of $\triangle ABC$ has centre I and touches the side BC at D. Let the midpoints of AD and BC be M and N respectively. Prove that M, I, N are collinear.

4. Points E, F on the sides CA, AB of $\triangle ABC$ are such that FE is parallel to BC; BE, CF intersect at X. Prove that AX is a median of $\triangle ABC$.

5. In $\triangle ABC$, AD is perpendicular to BC, prove that for any point P on AD we have
$$BP^2 - PC^2 = BD^2 - DC^2$$
and conversely, if P satisfies the above equation, then P lies on AD. Hence prove that the altitudes of a triangle are concurrent.

6. If H is the orthocentre of $\triangle ABC$ and AH produced meets BC at D and the circumcircle of $\triangle ABC$ at P, then prove that $HD = DP$.

7. Prove that if a point is taken anywhere on the circumcircle of an equilateral triangle, its distances from one of the vertices is equal to the sum of its distances from the remaining vertices.

8. In $\triangle ABC$, G is the centroid, prove that
$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

9. In $\triangle ABC$, if m_a, m_b, m_c are the medians then prove that
$$m_a + m_b + m_c < a + b + c < \frac{4}{3}(m_a + m_b + m_c).$$

10. In $\triangle ABC$, let AD be the internal angle bisector of $\angle A$, then show that
$$AD^2 = AB.AC - BD.DC.$$

11. If P, Q, R are points on the sides BC, CA, AB of $\triangle ABC$, such that the perpendiculars to the sides at these points are concurrent; then show that
$$BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2.$$

12. Let ABC be a triangle with unequal sides. The medians of $\triangle ABC$, when extended, intersect its circumcirle in points L, M, N. If L lies on the median through A and $LM = LN$, prove that
$$2BC^2 = CA^2 + AB^2.$$

13. If a, b, c are sides of a triangle then prove that
$$\frac{a}{c + a - b} + \frac{b}{a + b - c} + \frac{c}{b + c - a} \geq 3.$$

14. Let ABC be a triangle and h_a be the altitude through A. Prove that

$$(b + c)^2 \geq a^2 + 4h_a^2$$

where a, b, c represents the sides of the triangle ABC.

15. Suppose $ABCD$ is a cyclic quadrilateral inscribed in a circle of radius one unit. If

$$AB.BC.CD.DA \geq 4$$

then prove that $ABCD$ is a square.

The above problems are taken from [2], [3] and [4].

References

[1] D. Hilbert, *The Foundations of Geometry*, The Open Court Publishing Company, La Salle, Illinois, 1950.

[2] P. J. Mahanta, M. P. Saikia, *The Pursuit of Joy*, Vol 1, LAP Lambert Academic Publishing, Germany, 2011.

[3] M. R. Modak, S. A. Katre, V. V. Acharya, *An Excursion in Mathematics*, Bhaskaracharya Pratishthana, Pune, 2004.

[4] C. R. Pranesachar, B. J. Venkatachala, C. S. Yogananda, *Problem Primer for the Olympiad*, Prism Books, Bangalore, 2001.