Some Mathematical Problems[1]

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1. Prove that the sum of the degrees of a graph is twice the number of edges of the graph.

2. If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then find the number of factors of n.

3. What is the number of ways of tiling an $n \times 2$ board with dominoes?

4. Prove the following identities:
   (a) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.
   (b) $\sum_{m=k}^{n-k} \binom{m}{k}\binom{n-m}{k} = \binom{n+1}{2k+1}$.
   (c) $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$.
   (d) $\sum_{i=2}^{n} \binom{i}{2}\binom{n+2-i}{2} = \binom{n+3}{5}$.
   (e) $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$.

5. Prove that among any six people, there are always atleast three mutual friends or three mutual enemies.

6. Prove that $1.1! + 2.2! + 3.3! + \cdots + n.n! = (n+1)! - 1!$. **(Canada 196?)**

7. Let x be a real number, such that $x^{2014} - x^{2004}$ and $x^{2009} - x^{2004}$ are integers. Show that x is an integer. **(CMI 2014)**

8. How many functions are there from the set $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, n\}$? **(CMI 2014)**

9. Prove that
$$\sum_{i=0}^{k} (-1)^i 2^{k-i} \binom{i}{r}\binom{n}{k} = \binom{n}{k}.$$
(JEE 2003)

10. If n people are to be seated in n chairs marked from 1 to n, then what is the number of ways in which they can be seated so that the ith person is never seated at the ith marked chair?
(JEE Advanced 2014, JEE 1992)

11. Let A be a non empty finite sequence of n distinct integers, $a_1 < a_2 < \cdots < a_n$. Define $A + A = \{a_i + a_j \mid 1 \leq i, j \leq n\}$. Prove the following:
    (a) $\mid A + A \mid \geq 2n - 1$. **(CMI 2016)**
    (b) $\mid A + A \mid = 2n - 1$ if and only if A is an arithmetic progression. **(CMI 2016)**

[1]Some old Mathematical Olympiad style problem sheets and notes can be found here: http://homepage.univie.ac.at/manjil.saikia/math-olympiads/

12. Show that there are infinitely many perfect squares that can be written as the sum of six consecutive natural numbers. **(CMI 2011)**

13. Prove that

$$\sum_{i=2}^{n} \frac{i}{(i-2)! + (i-1)! + i!} = 1 - \frac{1}{n!}.$$

(CMI 2010)

14. Let a, b, c be natural numbers and $\frac{b}{a}$ be an integer. If a, b, c are in a geometric progression and the arithmetic mean of a, b, c is $b + 2$, then what is the value of $\frac{a^2 + a - 14}{a + 1}$?

(JEE Advanced 2014)

15. What is the co-efficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$?

(JEE Advanced 2014)

16. Prove that among any 51 numbers choosen from $\{1, 2, \ldots, 100\}$, there are always at least two which are relatively prime.

17. Let a_i be real numbers defined recursively as follows: $a_0 = 1$, and for all $n \geq 1$ we have

$$\sum_{k=0}^{n} \frac{a_n - k}{k + 1} = 0.$$

Show that $a_n > 0$ for all $n \geq 1$.

18. Evaluate

$$\sum_{k=1}^{n} \frac{k}{k^4 + k^2 + 1}.$$

19. Prove that the arithmetic mean of n real numbers is always as big as the geometric mean of those numbers. **(A.M.-G.M. Inequality)**

20. If a_i, b_i are real numbers for $1 \leq i \leq n$, then prove that

$$\left(\sum_{i=0}^{n} a_i b_i \right)^2 \leq \sum_{i=0}^{n} a_i^2 \sum_{i=0}^{n} b_i^2.$$

(Cauchy-Schwarz Inequality)

21. If x, y, z are real numbers, then prove that

$$\frac{1 + xy + yz}{(1 + y + z)^2} + \frac{1 + yz + yx}{(1 + z + x)^2} + \frac{1 + zx + zy}{(1 + x + y)^2} \geq 1.$$

22. If x, y, z are real numbers such that $xy + yz + zx = \frac{1}{3}$, then prove that

$$\frac{x}{x^2 - yz + 1} + \frac{y}{y^2 - zx + 1} + \frac{z}{z^2 - xy + 1} \geq \frac{1}{x + y + z}.$$

23. A positive integer is said to be multipowered number if in its prime factorization each prime factor has a power greater than or equal to 2. Prove that there are infinitely many pairs of adjacent positive numbers such that both are multipowered. **(Hungary ?)**

24. An alternating sign matrix is a square matrix with entries from the set $\{0, 1, -1\}$ such that all row and column sums are equal to 1, and that the non-zero entries alternate in sign. Prove that there can be only one non-zero entry in the first row of such a matrix. What about the last row? The first column? And the last column?

25. Evaluate

$$\cot\left(\sum_{n=0}^{23} \cot^{-1}(1 + \sum_{k=1}^{n} 2k)\right).$$

(JEE Advanced 2013)

26. Evaluate

$$\sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2.$$

(JEE Advanced 2013)

27. A pack contains n cards numbered from 1 to n. Two consecutively numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the number of the removed card is k, then which cards were removed? **(JEE Advanced 2013)**

28. Prove that the sum of 12 consecutive integers can never be a perfect square. **(ISI 2014)**

29. Let $A = \{1, 2, \ldots, n\}$ and $P = \{P(1), P(2), \ldots, P(n)\}$, then find the number of such P so that for all i, j in A, if $i < j < P(1)$ then $P(j) < P(i)$; and the number if $P(1) < i < j$ then $P(i) < P(j)$. **(ISI 2017)**

30. A class has 100 students, let a_i for $1 \leq i00$ denote the number of friends of the ith student. For each $0 \leq j9$, let c_j denote the number of students that have at least j friends. Show that

$$\sum_{i=1}^{100} a_i = \sum_{j=0}^{99} c_j.$$

(ISI 2014)

31. Find all positive integers n for which $5^n + 1$ is divisible by 7. **(ISI 2015)**

32. Let $g : \mathbb{N} \to \mathbb{N}$ with $g(n)$ being the product of the digits of n. Prove that $g(n) \leq n$ for all natural numbers n. **(ISI 2017)**

33. Evaluate

$$6 + \log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\cdots}}\right).$$

(JEE 2012)