# ENUMERATION OF DOMINO TILINGS OF AN AZTEC RECTANGLE WITH BOUNDARY DEFECTS

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Abstract. In this paper we enumerate domino tilings of an Aztec rectangle with arbitrary defects of size one on all boundary sides. This result extends previous work by different authors: Mills-Robbins-Rumsey and Elkies-Kuperberg-Larsen-Propp. We use the method of graphical condensation developed by Kuo and generalized by Ciucu, to prove our results; a common generalization of both Kuo's and Ciucu's result is also presented here.

## 1. Introduction

Elkies, Kuperberg, Larsen and Propp in their paper [3] introduced a new class of objects which they called Aztec diamonds. The Aztec diamond of order $n$ (denoted by AD($n$)) is the union of all unit squares inside the contour $|x|+|y|=n+1$ (see Figure 1 for an Aztec diamond of order 3). A domino is the union of any two unit squares sharing an edge, and a domino tiling of a region is a covering of the region by dominoes so that there are no gaps or overlaps. The authors in [3] and [4] considered the problem of counting domino tilings the Aztec diamond with dominoes and presented four different proofs of the following result.

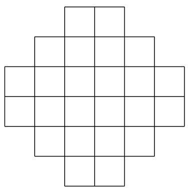

Figure 1. AD(3), the Aztec diamond of order 3.

**Theorem 1.1** (Elkies–Kuperberg–Larsen–Propp, [3, 4]). *The number of domino tilings of the Aztec diamond of order $n$ is $2^{n(n+1)/2}$.*

This work subsequently inspired lot of follow ups, including the natural extension of the Aztec diamond to the Aztec rectangle (see Figure 2). We denote by $\mathcal{AR}_{a,b}$ the Aztec rectangle which has $a$ unit squares on the southwestern side and $b$ unit squares on the northwestern side. In the remainder of this paper, we assume $b \geq a$ unless otherwise mentioned. For $a < b$, $\mathcal{AR}_{a,b}$ does not have any tiling by dominoes. The non-tileability of the region $\mathcal{AR}_{a,b}$ becomes evident if we

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look at the checkerboard representation of $\mathcal{AR}_{a,b}$ (see Figure 2). However, if we remove $b-a$ unit squares from the southeastern side then we have a simple product formula found by Mills, Robbins and Rumsey [7].

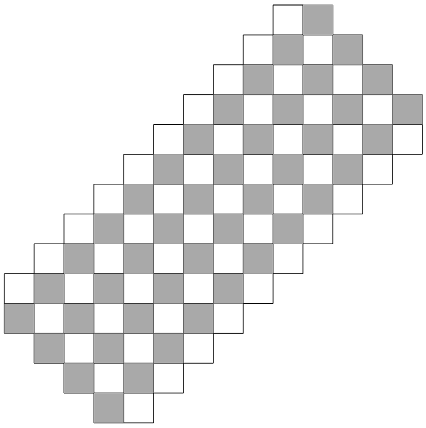


FIGURE 2. Checkerboard representation of an Aztec rectangle with $a=4, b=10$.

**Theorem 1.2** (Mills–Robbins–Rumsey, [7]). *Let $a<b$ be positive integers and $1 \leq s_1 < s_2 < \cdots < s_a \leq b$. Then the number of domino tilings of $\mathcal{AR}_{a,b}$ where all unit squares from the southeastern side are removed except for those in positions $s_1, s_2, \ldots, s_a$ is*

$$2^{a(a+1)/2} \prod_{1 \leq i < j \leq a} \frac{s_j - s_i}{j-i}.$$

The authors in [7] proved Theorem 1.2 in the context of monotone triangles, the connection between monotone triangles and Aztec diamonds was also used to derive a proof of Theorem 1.1 in [3]. Tri Lai [6] has recently generalized Theorem 1.2 to find a generating function, following the work of Elkies, Kuperberg, Larsen and Propp [3, 4]. Motivated by the recent work of Ciucu and Fischer [2], here we look at the problem of tiling an Aztec rectangle with dominoes if arbitrary unit squares are removed along the boundary of the Aztec rectangle.

This paper is structured as follows: in Section 2 we state our main results, in Section 3 we introduce our main tool in the proofs and present a slight generalization of it, in Section 4 we look at tilings of some special cases which are used in our main results. Finally, in Section 5 we prove the results described in Section 2 and illustrate one of our theorems with an example. The main ingredients in most of our proofs will be the method of condensation developed by Kuo [5] and its subsequent generalization by Ciucu [1].

## 2. Statements of Main Results

In order to create a region that can be tiled by dominoes we have to remove $b-a$ (henceforth denoted by $k$) more white squares than black squares along the boundary of $\mathcal{AR}_{a,b}$. There are $2b$ white squares and $2a$ black squares on the boundary of $\mathcal{AR}_{a,b}$. We choose $n+k$ of the white squares that share an edge with the boundary and denote them by $\beta_1, \beta_2, \dots, \beta_{n+k}$ (we will refer to them as defects of type $\beta$). We choose any $n$ squares from the black squares which share an edge with the boundary and denote them by $\alpha_1, \alpha_2, \dots, \alpha_n$ (we refer to them as defects of type $\alpha$). We consider regions of the type $\mathcal{AR}_{a,b} \setminus \{\beta_1, \dots, \beta_{n+k}, \alpha_1, \dots, \alpha_n\}$, which are more general than the type considered in [3, 7].

It is also known that domino tilings of a region can be identified with perfect matchings[1] of its planar dual graph[2], so for any region $R$ on the square lattice we denote by $\mathrm{M}(R)$ the number of domino tilings of $R$. We now state the main results of this paper below. The first result is concerned with the case when the defects are confined to three of the four sides of the Aztec rectangle (defects do not occur on one of the sides with shorter length), and provides a Pfaffian expression for the number of tilings of such a region, with each entry in the Pfaffian (see Definition 2.1 below) being given by a simple product or by a sum or product of quotients of factorials and powers of 2. The second result gives a nested Pfaffian expression for the general case when we do not restrict the occurrence of defects on any boundary side. The third result deals with the case of an Aztec diamond with arbitrary defects on the boundary and gives a Pfaffian expression for the number of tilings of such a region, with each entry in the Pfaffian being given by a simple sum of quotients of factorials and powers of 2.

**Definition 2.1.** *Let $A = (a_{i,j})$ be a $2n \times 2n$ antisymmetric matrix and $\Gamma_n$ be the set of all perfect matchings of $K_{2n}$. Then the Pfaffian of $A$ (denoted $\mathrm{Pf}(A)$) is defined as*

$$\mathrm{Pf}(A) = \sum_{\pi=\{(i_1,j_1),\dots,(i_n,j_n)\}\in\Gamma_n} \operatorname{sgn}\pi \prod_{k=1}^{n} a_{i_k,j_k}$$

*where $\operatorname{sgn}\pi = \operatorname{sgn} i_1 j_1 i_2 j_2 \dots i_n j_n$. There are many ways to write $\pi$, so to see that $\mathrm{Pf}(A)$ is well-defined we assume that $i_k < j_k$ and $i_1 < i_2 < \cdots < i_n$.*

We define the region $\mathcal{AR}_{a,b}^k$ to be the region obtained from $\mathcal{AR}_{a.b}$ by adding a string of $k$ unit squares along the boundary of the southeastern side as shown in Figure 3. We denote this string of $k$ unit squares by $\gamma_1, \gamma_2, \dots, \gamma_k$ and refer to them as defects of type $\gamma$.

**Theorem 2.2.** *Assume that one of the two sides on which defects of type $\alpha$ can occur does not actually have any defects on it. Without loss of generality, we assume this to be the southwestern side. Let $\delta_1, \dots, \delta_{2n+2k}$ be the elements of the set $\{\beta_1, \dots, \beta_{n+k}\} \cup \{\alpha_1, \dots, \alpha_n\} \cup \{\gamma_1, \dots, \gamma_k\}$ listed in a cyclic order.*

*Then we have*

(2.1)

$$\mathrm{M}(\mathcal{AR}_{a,b}\backslash\{\beta_1, \dots, \beta_{n+k}, \alpha_1, \dots, \alpha_n\}) = \frac{1}{[\mathrm{M}(\mathcal{AR}_{a,b}^k)]^{n+k-1}} \mathrm{Pf}[(\mathrm{M}(\mathcal{AR}_{a,b}^k\backslash\{\delta_i, \delta_j\}))_{1\le i<j\le 2n+2k}],$$

*where all the terms on the right hand side are given by explicit formulas:*

*(1) $\mathrm{M}(\mathcal{AR}_{a,b}^k)$ is given by Theorem 1.1,*

[1]A perfect matching of a graph is a matching in which every vertex of the graph is incident to exactly one edge of the matching.

[2]By the planar dual graph, we mean the graph that is obtained if we identify each unit square with a vertex and unit squares sharing an edge with each other is identified with an edge.

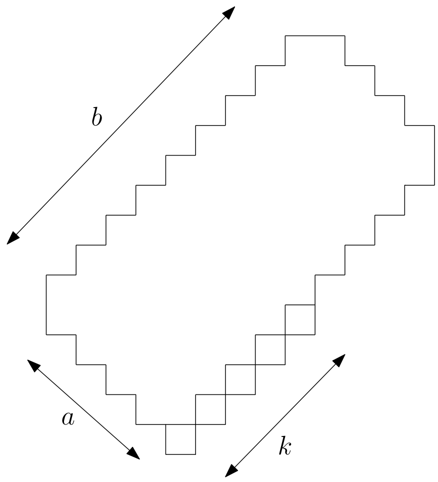



FIGURE 3. $\mathcal{AR}_{a,b}^k$ with $a = 4, b = 9, k = 5$.

*(2)* $\mathrm{M}(\mathcal{AR}_{a,b}^k \setminus \{\beta_i, \alpha_j\})$ *is given by Proposition 4.11 if* $\beta_i$ *is on the southeastern side and not above a* $\gamma$ *defect; otherwise it is* $0$,

*(3)* $\mathrm{M}(\mathcal{AR}_{a,b}^k \setminus \{\beta_i, \gamma_j\})$ *is given by Theorem 1.1 if* $\beta_i$ *is above a* $\gamma$ *defect; it is given by Proposition 4.10 if the* $\beta$*-defect is in the northwestern side and its distance from the western corner is more than the distance of the* $\gamma$*-defect from the southern corner; it is given by Propositions 4.3 if the* $\beta$ *defect is on the southeastern side; otherwise it is* $0$,

*(4)* $\mathrm{M}(\mathcal{AR}_{a,b}^k \setminus \{\beta_i, \beta_j\}) = \mathrm{M}(\mathcal{AR}_{a,b}^k \setminus \{\alpha_i, \alpha_j\}) = \mathrm{M}(\mathcal{AR}_{a,b}^k \setminus \{\alpha_i, \gamma_j\}) = \mathrm{M}(\mathcal{AR}_{a,b}^k \setminus \{\gamma_i, \gamma_j\}) = 0$.

**Theorem 2.3.** *Let* $\beta_1, \ldots, \beta_{n+k}$ *be arbitrary defects of type* $\beta$ *and* $\alpha_1, \ldots, \alpha_n$ *be arbitrary defects of type* $\alpha$ *along the boundary of* $\mathcal{AR}_{a,b}$. *Then* $\mathrm{M}(\mathcal{AR}_{a,b} \setminus \{\beta_1, \ldots, \beta_{n+k}, \alpha_1, \ldots, \alpha_n\})$ *is equal to the Pfaffian of a* $2n \times 2n$ *matrix whose entries are Pfaffians of* $(2k+2) \times (2k+2)$ *matrices of the type in the statement of Theorem 2.2.*

In the special case when the number of defects of both types are the same, that is when $k = 0$ we get an Aztec diamond with arbitrary defects on the boundary and the number of tilings can be given by a Pfaffian where the entries of the Pfaffian are explicit, as stated in the theorem below.

**Theorem 2.4.** *Let* $\beta_1, \ldots, \beta_n$ *be arbitrary defects of type* $\beta$ *and* $\alpha_1, \ldots, \alpha_n$ *be arbitrary defects of type* $\alpha$ *along the boundary of* $\mathrm{AD}(a)$, *and let* $\delta_1, \ldots, \delta_{2n}$ *be a cyclic listing of the elements of the set* $\{\beta_1, \ldots, \beta_n\} \cup \{\alpha_1, \ldots, \alpha_n\}$. *Then*

$$\mathrm{M}(\mathrm{AD}(a) \setminus \{\beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_n\}) = \frac{1}{[\mathrm{M}(\mathrm{AD}(a))]^{n-1}} \mathrm{Pf}[(\mathrm{M}(\mathrm{AD}(a) \setminus \{\delta_i, \delta_j\}))_{1 \le i < j \le 2n}], \tag{2.2}$$

*where the values of* $\mathrm{M}(\mathrm{AD}(a) \setminus \{\delta_i, \delta_j\}))$ *are given explicitly as follows:*

*(1)* $\mathrm{M}(\mathrm{AD}(a) \setminus \{\beta_i, \alpha_j\}))$ *is given by Proposition 4.11,*

*(2)* $\mathrm{M}(\mathrm{AD}(a) \setminus \{\beta_i, \beta_j\})) = \mathrm{M}(\mathrm{AD}(a) \setminus \{\alpha_i, \alpha_j\})) = 0$.

## 3. A result on Graphical Condensation

The proofs of our main results are based on Ciucu's generalization [1] of Kuo's graphical condensation [5] which we state below. The aim of this section is also to present our small generalization of Ciucu's result.

Let $G$ be a weighted graph, where the weights are associated with each edge of $G$, and let $\mathrm{M}(G)$ denote the sum of the weights of the perfect matchings of $G$, where the weight of a perfect matching is taken to be the product of the weights of its constituent edges. We are interested in graphs with edge weights all equaling 1, which corresponds to tilings of the region in our special case.

**Theorem 3.1** (Ciucu, [1]). *Let $G$ be a planar graph with the vertices $a_1, a_2, \dots, a_{2k}$ appearing in that cyclic order on a face of $G$. Consider the skew-symmetric matrix $A = (a_{ij})_{1 \leq i,j \leq 2k}$ with entries given by*

$$a_{ij} := \mathrm{M}(G \setminus \{a_i, a_j\}), \text{ if } i < j. \tag{3.1}$$

*Then we have that*

$$\mathrm{M}(G \setminus \{a_1, a_2, \dots, a_{2k}\}) = \frac{\mathrm{Pf}(A)}{[\mathrm{M}(G)]^{k-1}}. \tag{3.2}$$

Although Theorem 3.1 is enough for our purposes, we state and prove a slightly more general version of the theorem below. It turns out that our result is a common generalization for the condensation results in [5] as well as Theorem 3.1 which follows immediately from Theorem 3.2 below if we consider $a_1, \dots, a_{2k} \in \mathrm{V}(G)$. We also mention that Corollary 3.3 of Theorem 3.2, does not follow from Theorem 3.1.

To state and prove our result, we will need to make some notations and concepts clear. We consider the symmetric difference on the vertices and edges of a graph. Let $H$ be a planar graph and $G$ be an induced subgraph of $H$ and let $W \subseteq \mathrm{V}(H)$. Then we define $G + W$ to be the induced subgraph of $H$ with vertex set $\mathrm{V}(G + W) = \mathrm{V}(G) \Delta \mathrm{V}(W)$, where $\Delta$ denotes the symmetric difference of sets. Now we are in a position to state our result below.

**Theorem 3.2.** *Let $H$ be a planar graph and let $G$ be an induced subgraph of $H$ with the vertices $a_1, a_2, \dots, a_{2k}$ appearing in that cyclic order on a face of $H$. Consider the skew-symmetric matrix $A = (a_{ij})_{1 \leq i,j \leq 2k}$ with entries given by*

$$a_{ij} := \mathrm{M}(G + \{a_i, a_j\}), \text{ if } i < j. \tag{3.3}$$

*Then we have that*

$$\mathrm{M}(G + \{a_1, a_2, \dots, a_{2k}\}) = \frac{\mathrm{Pf}(A)}{[\mathrm{M}(G)]^{k-1}}. \tag{3.4}$$

**Corollary 3.3.** [5, Theorem 2.4] *Let $G = (V_1, V_2, E)$ be a bipartite planar graph with $|V_1| = |V_2| + 1$; and let $w, x, y$ and $z$ be vertices of $G$ that appear in cyclic order on a face of $G$. If $w, x, y \in V_1$ and $z \in V_2$ then*

$$\mathrm{M}(G - \{w\})\,\mathrm{M}(G - \{x, y, z\}) + \mathrm{M}(G - \{y\})\,\mathrm{M}(G - \{w, x, z\}) = \mathrm{M}(G - \{x\})\,\mathrm{M}(G - \{w, y, z\}).$$

*Proof.* Take $n = 2$, $a_1 = w, a_2 = x, a_3 = y, a_4 = z$ and $G = H \setminus \{a_1\}$ in Theorem 3.2. □

The proof of Theorem 3.1 follows from the use of some auxiliary results. In the vein of those results, we need the following proposition to complete our proof of Theorem 3.2.

**Proposition 3.4.** *Let $H$ be a planar graph and $G$ be an induced subgraph of $H$ with the vertices $a_1, \dots, a_{2k}$ appearing in that cyclic order among the vertices of some face of $H$. Then*

$$\begin{aligned} &\mathrm{M}(G)\,\mathrm{M}(G+\{a_1,\dots,a_{2k}\}) + \sum_{l=2}^{k} \mathrm{M}(G+\{a_1,a_{2l-1}\})\,\mathrm{M}(G+\overline{\{a_1,a_{2l-1}\}}) \\ &\qquad = \sum_{l=1}^{k} \mathrm{M}(G+\{a_1,a_{2l}\})\,\mathrm{M}(G+\overline{\{a_1,a_{2l}\}}), \end{aligned} \tag{3.5}$$

*where $\overline{\{a_i,a_j\}}$ stands for the complement of $\{a_i,a_j\}$ in the set $\{a_1,\dots,a_{2k}\}$.*

Our proof follows closely that of the proof of an analogous proposition given by Ciucu [1, Proposition 1] with very little difference and hence we refer to it for the sake of brevity.

*Proof.* We recast equation (3.5) in terms of disjoint unions of cartesian products as follows

$$\begin{aligned} &\mathcal{M}(G)\times\mathcal{M}(G+\{a_1,\dots,a_{2k}\}) \cup \mathcal{M}(G+\{a_1,a_3\})\times\mathcal{M}(G+\overline{\{a_1,a_3\}}) \cup \dots \\ &\qquad \cup\mathcal{M}(G+\{a_1,a_{2k-1}\})\times\mathcal{M}(G+\overline{\{a_1,a_{2k-1}\}}) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\mathcal{M}(G+\{a_1,a_2\})\times\mathcal{M}(G+\overline{\{a_1,a_2\}}) \cup \mathcal{M}(G+\{a_1,a_4\})\times\mathcal{M}(G+\overline{\{a_1,a_4\}}) \cup \dots \\ &\qquad \cup\mathcal{M}(G+\{a_1,a_{2k}\})\times\mathcal{M}(G+\overline{\{a_1,a_{2k}\}})\cup \end{aligned} \tag{3.7}$$

where $\mathcal{M}(F)$ denotes the set of perfect matchings of the graph $F$. For each element $(\mu,\nu)$ of (3.6) or (3.7), we think of the edges of $\mu$ as being marked by solid lines and that of $\nu$ as being marked by dotted lines, on the same copy of the graph $H$. If there are any edges common to both then we mark them with both solid and dotted lines.

We now define the weight of $(\mu,\nu)$ to be the product of the weight of $\mu$ and the weight of $\nu$. Thus, the total weight of the elements in the set (3.6) is same as the left hand side of equation (3.5) and the total weight of the elements in the set (3.7) equals the right hand side of equation (3.5). To prove our result, we have to construct a weight-preserving bijection between the sets (3.6) and (3.7). The construction is similar to the one given by Ciucu [1, Proposition 1], so we mention only the essential details below and refer the reader to Ciucu's proof.

Let $(\mu,\nu)$ be an element in (3.6). Then we have two possibilities as discussed in the following. If $(\mu,\nu)\in\mathcal{M}(G)\times\mathcal{M}(G+\{a_1,\dots,a_{2k}\})$, we consider the path containing $a_1$ and change a solid edge to a dotted edge and a dotted edge to a solid edge in order to obtain a new pair of matchings. Let this pair of matchings be $(\mu',\nu')$. For a clearer view of this *shifting along the path* process, we refer the reader to Ciucu's proof. The path we have obtained must connect $a_1$ to one of the even-indexed vertices. So $(\mu',\nu')$ is an element of (3.7).

If $(\mu,\nu)\in\mathcal{M}(G+\{a_1,a_3\})\times\mathcal{M}(G+\overline{\{a_1,a_3\}})$, then we map it to a pair of matchings $(\mu',\nu')$ obtained by reversing the solid and dotted edges along the path containing $a_3$. With a similar reasoning like above, this path must connect $a_3$ to one of the even-indexed vertices and a similar argument will show that indeed $(\mu',\nu')$ is an element of (3.7). If $(\mu,\nu)\in\mathcal{M}(G+\{a_1,a_{2i+1}\})\times\mathcal{M}(G+\overline{\{a_1,a_{2i+1}\}})$ with $i>1$, we have the same construction with $a_3$ replaced by $a_{2i+1}$.

The map $(\mu,\nu)\mapsto(\mu',\nu')$ is invertible because given an element in $(\mu',\nu')$ of (3.7), the pair $(\mu,\nu)$ that is mapped to it is obtained by shifting along the path that contains the vertex $a_{2i}$, such that $(\mu',\nu')\in\mathcal{M}(G+\{a_1,a_{2i}\})\times\mathcal{M}(G+\overline{\{a_1,a_{2i}\}})$. The map we have defined is weight-preserving and this proves the proposition. □

Now we can prove Theorem 3.2, which is essentially the same proof as that of Theorem 3.1, but now uses our more general Proposition 3.4.

*Proof of Theorem 3.2.* We prove the statement by induction on $k$. For $k = 1$ it follows from the fact that

$$\operatorname{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a.$$

For the induction step, we assume that the statement holds for $k-1$ with $k \geq 2$. Let $A$ be the matrix

$$\begin{pmatrix} 0 & \mathrm{M}(G+\{a_1,a_2\}) & \mathrm{M}(G+\{a_1,a_3\}) & \cdots & \mathrm{M}(G+\{a_1,a_{2k}\}) \\ -\mathrm{M}(G+\{a_1,a_2\}) & 0 & \mathrm{M}(G+\{a_2,a_3\}) & \cdots & \mathrm{M}(G+\{a_2,a_{2k}\}) \\ -\mathrm{M}(G+\{a_1,a_3\}) & -\mathrm{M}(G+\{a_2,a_3\}) & 0 & \cdots & \mathrm{M}(G+\{a_3,a_{2k}\}) \\ \vdots & \vdots & \vdots & & \vdots \\ -\mathrm{M}(G+\{a_1,a_{2k}\}) & -\mathrm{M}(G+\{a_2,a_{2k}\}) & -\mathrm{M}(G+\{a_3,a_{2k}\}) & \cdots & 0 \end{pmatrix}.$$

By a well-known property of Pfaffians, we have

$$\operatorname{Pf}(A) = \sum_{i=2}^{2k} (-1)^i \,\mathrm{M}(G+\{a_1,a_i\}) \operatorname{Pf}(A_{1i}). \tag{3.8}$$

Now, the induction hypothesis applied to the graph $G$ and the $2k-2$ vertices in $\overline{\{a_i,a_j\}}$ gives us

$$[\mathrm{M}(G)]^{k-2}\,\mathrm{M}(G+\overline{\{a_1,a_i\}}) = \operatorname{Pf}(A_{1i}), \tag{3.9}$$

where $A_{1i}$ is same as in equation (3.8). So using equations (3.8) and (3.9) we get

$$\operatorname{Pf}(A) = [\mathrm{M}(g)]^{k-2} \sum_{i=2} 2k(-1)^i \,\mathrm{M}(G+\{a_1,a_i\})\,\mathrm{M}(G+\overline{\{a_1,a_i\}}). \tag{3.10}$$

Now using Proposition 3.4, we see that the above sum is $\mathrm{M}(G)\,\mathrm{M}(G+\{a_1,\dots,a_{2k}\})$ and hence equation (3.10) implies (3.4).

□

## 4. Some family of regions with defects

In this section, we find the number of dominoe tilings of certain regions which appear in the statement of Theorem 2.2 and Theorem 2.4. We define the binomial coefficients that appear in this section as follows

$$\binom{c}{d} := \begin{cases} \dfrac{c(c-1)\cdots(c-d+1)}{d!}, & \text{if } d \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Our formulas also involve hypergeometric series. We recall that the hypergeometric series of parameters $a_1,\dots,a_r$ and $b_1,\dots,b_s$ is defined as

$${}_rF_s\left[\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix};z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_r)_k}{(b_1)_k\cdots(b_s)_k}\frac{z^k}{k!},$$

where the Pochhammer symbol is defined as $(n)_m := n(n+1)(n+2)\cdots(n+m-1)$. Most of the proofs are quite similar and hence we prove only Propositions 4.3 and 4.11 and omit certain details from some of the proofs of the other propositions in this section.

We also fix a notation for the remainder of this paper as follows, if we remove the squares labelled $2, 4, 7$ from the southeastern boundary of $\mathcal{AR}_{4,7}$, we denote it by $\mathcal{AR}_{4,7}(2, 4, 7)$. In the derivation of the results in this section, the following two corollaries of Theorem 1.2 will be used.

**Corollary 4.1.** *The number of tilings of* $\mathcal{AR}_{a,a+1}(i)$ *is given by*

$$2^{a(a+1)/2}\binom{a}{i-1}.$$

**Corollary 4.2.** *The number of tilings of* $\mathcal{AR}_{a,b}(2, \ldots, b-a+1)$ *is given by*

$$2^{a(a+1)/2}\binom{b-1}{a-1}.$$

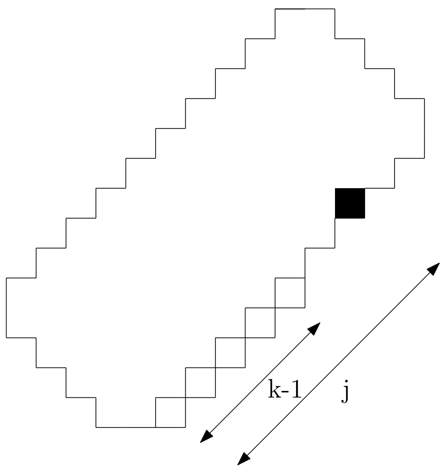



FIGURE 4. An Aztec rectangle with $k-1$ squares added on the southeastern side and a defect on the $j$-th position shaded in black; here $a = 4, b = 10, k = 6, j = 8$.

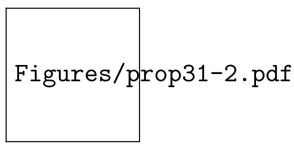



FIGURE 5. An Aztec rectangle with $k-1$ squares added on the southeastern side and a defect on the $j$-th position shaded in black; here $a = 4, b = 10, k = 6, j = 3$.

**Proposition 4.3.** *Let $1 \leq a \leq b$ be positive integers with $k = b - a > 0$, then the number of domino tilings of $\mathcal{AR}_{a,b}(j)$ with $k-1$ squares added to the southeastern side starting at the second position (and not at the bottom) as shown in the Figure 4 for $j \geq k$ is given by*

$$2^{a(a+1)/2}\binom{a+k-1}{j-1}\binom{j-2}{k-1}\,{}_3F_2\left[\begin{matrix}1, 1-j, 1-k\\ 2-j, 1-a-k\end{matrix};1\right]. \tag{4.1}$$

*If $j \leq k-1$, then the number of tilings of $\mathcal{AR}_{a,b}(j)$ with $k-1$ squares added to the southeastern side starting at the second position (and not at the bottom) as shown in the Figure 5 is given by*

$$2^{a(a+1)/2}.$$

*Proof.* Let us denote the region in Figure 4 by $\mathcal{AR}_{a,b}^{k-1,j}$ and we work with the planar dual graph of the region $\mathcal{AR}_{a,b}^{k-1,j}$ and count the number of matchings of that graph.

We first deal with the case when $j \geq k$. We notice that the first added square in any domino tiling of the region in Figure 4 by dominoes has two possibilities to be matched up with squares marked in grey in the Figure 6. This observation allows us to write the number of tilings of $\mathcal{AR}_{a,b}^{k-1,j}$ in terms of the following recursion

$$\mathrm{M}(\mathcal{AR}_{a,b}^{k-1,j}) = \mathrm{M}(\mathcal{AR}_{a,b-1}^{k-2,j-1}) + \mathrm{M}(\mathcal{AR}_{a,b}(2,3,\dots,k,j)). \tag{4.2}$$

which can be verified from Figure 7.

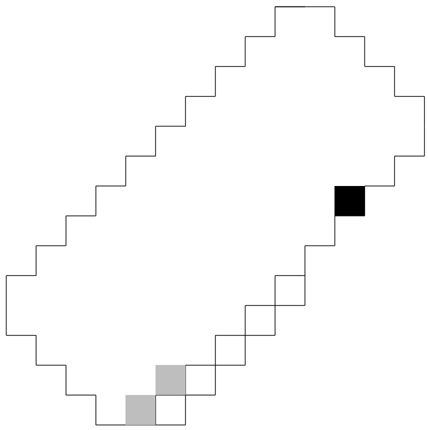


FIGURE 6. $\mathcal{AR}_{a,b}^{k-1,j}$ with the possible choices for the first added square in a tiling; here $a = 4, b = 10, k = 6, j = 8$.

We can now continue to descend further in a similar way until we have no added squares on the southeastern side remaining. Thus, we can repeat this process for $\mathcal{AR}_{a,b-1}^{k-2,j-1}$, then for $\mathcal{AR}_{a,b-2}^{k-3,j-2}$ and so on. Repeatedly using equation (4.2) as a template for this descend process $k-1$ times successively, we shall finally obtain

$$\mathrm{M}(\mathcal{AR}_{a,b}^{k-1,j}) = \sum_{l=0}^{k-2}\mathrm{M}(\mathcal{AR}_{a,b-l}(2,3,\dots,k-l,j-l)) + \mathrm{M}(\mathcal{AR}_{a,a+1}(j-k+1)). \tag{4.3}$$

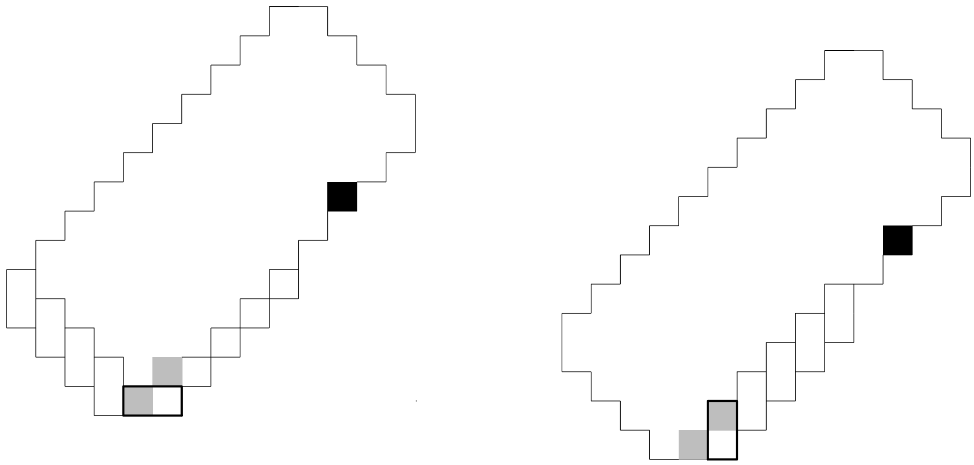


FIGURE 7. Choices for the tilings of $\mathcal{AR}_{a,b}^{k-1,j}$ with forced dominoes; here $a = 4, b = 10, k = 6, j = 8$.

Now, plugging in the values of the quantities in the right hand side of equation (4.3) from Theorem 1.2 and Corollary 4.1 we shall obtain the following.

$$\begin{aligned} \mathrm{M}(\mathcal{AR}_{a,b}^{k-1,j}) =& \sum_{l=0}^{k-2} 2^{a(a+1)/2} \binom{a+k-l-1}{a+k-j} \binom{j-l-2}{k-l-1} + 2^{a(a+1)/2} \binom{a}{a+k-j} \\ =& 2^{a(a+1)/2} \sum_{l=0}^{k-1} \binom{a+k-l-1}{a+k-j} \binom{j-l-2}{k-l-1}. \end{aligned} \tag{4.4}$$

Using standard techniques, if we transform the above binomial sum into hypergeometric notation, then we shall obtain equation (4.1).

Figures/prop31-3.pdf

FIGURE 8. $\mathcal{AR}_{a,b}^{k-1,j}$, when $j \leq k-1$; here $a = 4, b = 10, k = 6, j = 3$.

For the case when $j \leq k-1$, we see that there are many forced dominoes in any tiling (see Figure 8) and the region we want to tile is reduced to an Aztec diamond of order $a$, and this completes the proof.

□

One of the main ingredients in our proofs of the remaining results in this section is the following results of Kuo [5].

**Theorem 4.4.** [5, Theorem 2.3] *Let $G = (V_1, V_2, E)$ be a plane bipartite graph in which $|V_1| = |V_2|$. Let $w, x, y$ and $z$ be vertices of $G$ that appear in cyclic order on a face of $G$. If $w, x \in V_1$ and $y, z \in V_2$ then*

$$\mathrm{M}(G - \{w, z\})\,\mathrm{M}(G - \{x, y\}) = \mathrm{M}(G)\,\mathrm{M}(G - \{w, x, y, z\}) + \mathrm{M}(G - \{w, y\})\,\mathrm{M}(G - \{x, z\}).$$

**Theorem 4.5.** [5, Theorem 2.5] *Let $G = (V_1, V_2, E)$ be a plane bipartite graph in which $|V_1| = |V_2|+2$. Let the vertices $w, x, y$ and $z$ appear in that cyclic order on a face of $G$. Let $w, x, y, z \in V_1$, then*

$$\mathrm{M}(G - \{w, y\})\,\mathrm{M}(G - \{x, z\}) = \mathrm{M}(G - \{w, x\})\,\mathrm{M}(G - \{y, z\}) + \mathrm{M}(G - \{w, z\})\,\mathrm{M}(G - \{x, y\}).$$

**Theorem 4.6.** [5, Theorem 2.1] *Let $G = (V_1, V_2, E)$ be a plane bipartite graph with $|V_1| = |V_2|$ and $w, x, y, z$ be vertices of $G$ that appear in cyclic order on a face of $G$. If $w, y \in V_1$ and $x, z \in V_2$ then*

$$\mathrm{M}(G)\,\mathrm{M}(G - \{w, x, y, z\}) = \mathrm{M}(G - \{w, x\})\,\mathrm{M}(G - \{y, z\}) + \mathrm{M}(G - \{w, z\})\,\mathrm{M}(G - \{x, y\}).$$

The following proposition does not appear explicitly anywhere, but it is used implicitly in deriving Proposition 4.9.

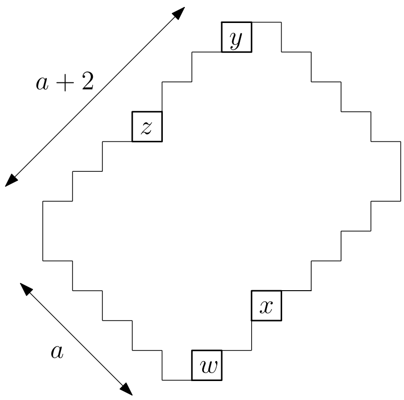



FIGURE 9. An $a \times (a + 2)$ Aztec rectangle with some labelled squares; here $a = 5$.

**Proposition 4.7.** *Let $1 \leq a$ be a positive integer, then the number of tilings of $\mathcal{AR}_{a,a+2}$ with a defect at the $i$-th position on the southeastern side counted from the south corner and a defect on the $j$-th position on the northwestern side counted from the west corner is given by*

$$2^{a(a+1)/2}\left[\binom{a}{i-2}\binom{a}{j-1} + \binom{a}{i-1}\binom{a}{j-2}\right]. \tag{4.5}$$

*Proof.* If $j = 1$ or $j = a + 2$, then the region we want to tile reduces to the type in Theorem 1.2 due to forced dominoes in any tiling, as can be seen from Figure 10 for the case when $j = 1$. The case when $j = a + 2$ is similar. We do not worry about the case when both $i$ and $j$ equals 1 or $a + 2$ because then the expression in (4.5) is 0. It is easy to see that the expression (4.5)

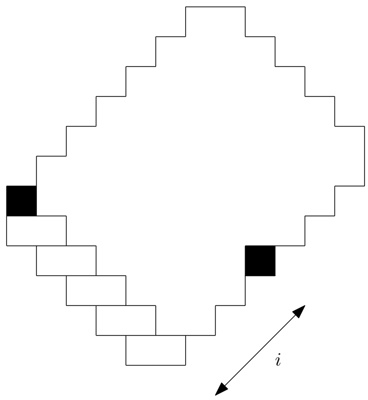



FIGURE 10. An $a \times (a+2)$ Aztec rectangle with defects marked; here $a = 5$, $i = 1$.

is satisfied in all of the other cases. By symmetry, this also takes care of the cases $i = 1$ and $i = a + 2$.

In the rest of the proof, we now assume that $1 < i, j < a + 2$ and let us denote the region we are interested in by $\mathrm{O}(a)_{i,j}$. We now use Theorem 4.5 with the vertices as indicated in Figure 9 to obtain the following identity (Figure 11).

$$\begin{aligned} \mathrm{M}(\mathrm{AD}(a))\,\mathrm{M}(\mathrm{O}(a)_{i,j}) = &\,\mathrm{M}(\mathcal{AR}_{a,a+1}(i-1))\,\mathrm{M}(\mathcal{AR}_{a,a+1}(j)) \\ &+ \mathrm{M}(\mathcal{AR}_{a,a+1}(j-1))\,\mathrm{M}(\mathcal{AR}_{a,a+1}(i)). \end{aligned} \tag{4.6}$$

Now, using Theorem 1.1 and Corollary 4.1 in equation (4.6) we get (4.5).

□

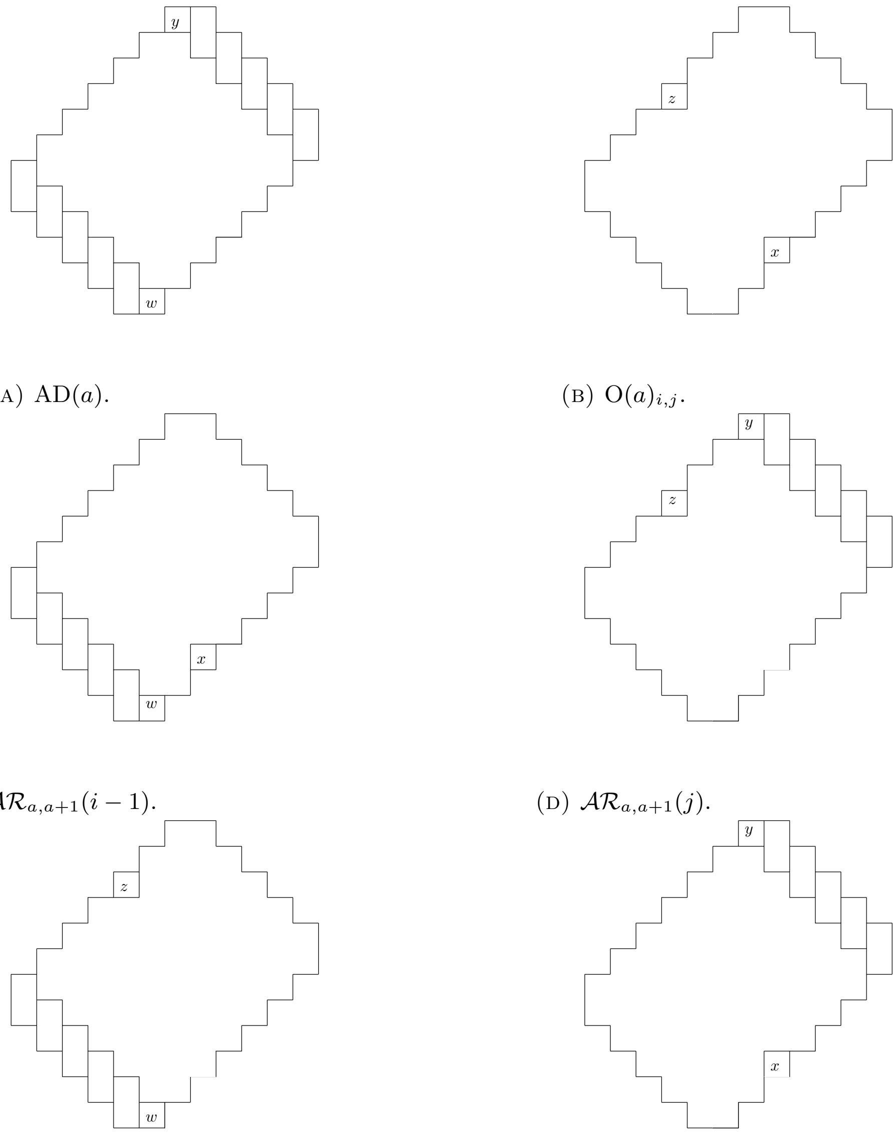
*Remark* 4.8. Ciucu and Fischer [2] have a similar result for the number of lozenge tilings of a hexagon with defects on opposite sides (Proposition 4 in their paper). They also make use of Kuo's condensation result, Theorem 4.4 and obtain the following identity

$$\begin{aligned} &\mathrm{OPP}(a,b,c)_{i,j}\,\mathrm{OPP}(a-2,b,c)_{i-1,j-1} \\ &\quad = \mathrm{OPP}(a-1,b,c)_{i-1,j-1}\,\mathrm{OPP}(a-1,b,c)_{i,j} \\ &\qquad - \mathrm{OPP}(a-1,b-1,c+1)_{i,j-1}\,\mathrm{OPP}(a-1,b+1,c-1)_{i-1,j} \end{aligned}$$

where $\mathrm{OPP}(a,b,c)_{i,j}$ denotes the number of lozenge tilings of a hexagon $H_{a,b,c}$ with opposite side lengths $a, b, c$ and with two defects in positions $i$ and $j$ on opposite sides of length $a$, where $a, b, c, i, j$ are positive integers with $1 \leq i, j \leq a$.

In their use of Kuo's result, they take the graph $G$ to be the planar dual graph of $H_{a,b,c}$ with two defects in positions $i$ and $j$ on opposite sides of length $a$ (the resulting number of lozenge tilings of this region is then $\mathrm{OPP}(a,b,c)_{i,j}$), but if we take the graph $G$ to be the planar dual graph of $H_{a,b,c}$ and use Theorem 4.4 with an appropriate choice of labels, we get the following identity

$$\begin{aligned} \mathrm{OPP}(a,b,c)_{i,j}\,\mathrm{H}(a-1,b,c) = &\,\mathrm{H}(a,b,c)\,\mathrm{OPP}(a-1,b,c)_{i,j} \\ &+ \mathrm{H}(a,c-1,b+1,a-1,c,b)_i\,\mathrm{H}(a,c,b,a-1,c+1,b-1)_{a-j+1} \end{aligned}$$




(A) AD$(a)$. (B) O$(a)_{i,j}$.

(C) $\mathcal{AR}_{a,a+1}(i-1)$. (D) $\mathcal{AR}_{a,a+1}(j)$.

(E) $\mathcal{AR}_{a,a+1}(j-1)$. (F) $\mathcal{AR}_{a,a+1}(i)$.

FIGURE 11. Some forced dominoes in the proof of Proposition 4.7 where the vertices we remove are labelled.

where $\mathrm{H}(a,b,c)$ denotes the number of lozenge tilings of the hexagon with opposite sides of length $a,b,c$ and $\mathrm{H}(m,n,o,p,q,r)_k$ denotes the number of lozenge tilings of a hexagon with side lengths $m,n,o,p,q,r$ with a defect at position $k$ on the side of length $m$. Then, Proposition 4 of Ciucu and Fischer [2] follows more easily without the need for contiguous relations of hypergeometric series that they use in their paper.

The following result does not appear explicitly in the statements of our main theorems, but this result is essential in deriving Proposition 4.10 later.

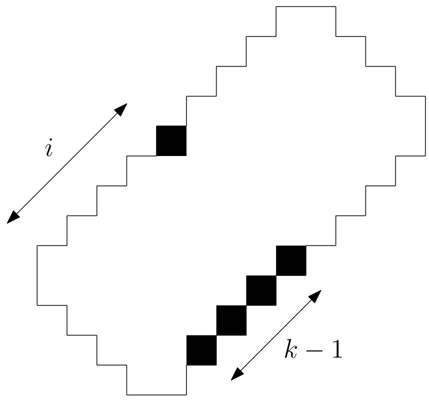



FIGURE 12. An $a \times b$ Aztec rectangle with defects marked in black; here $a = 4, b = 9, k = 5, i = 5$.

**Proposition 4.9.** *Let $1 \leq a, i \leq b$ be positive integers with $k = b - a > 0$ and $y = \min\{i, k\}$, then the number of domino tilings of $\mathcal{AR}_{a,b}(2, 3, \ldots, k)$ with a defect on the northwestern side in the $i$-th position counted from the west corner as shown in the Figure 12 is given by*

$$2^{a(a+1)/2}\binom{a+y-2}{a-1}\binom{a}{i-y}\,{}_3F_2\left[\begin{matrix}1, 1-y, i-a-y\\ i-y+1, 2-a-y\end{matrix}; -1\right].$$

*Proof.* Our proof will be by induction on $b = a + k$. The base case of induction will follow if we verify the result for $a = 2, k = 1$ in which case $b = 3$. To check our base case it is now enough to verify the formula for $a = 2, k = 1, i = 1, 2, 3$, in which case $y = 1$. This is easily verified as when $i = 1$, we have many forced dominoes and we get the region shown in Figure 13, which is AD(2). When $i = 2$, we see that the region we obtain is of the type as described in Corollary 4.1 and finally when $i = 3$, then we get a region of the type in Theorem 1.2. It is easily verified that in all these cases, the regions satisfy the statement of the result.

We now deal with the case when $b > 3$, and $i = 1$ or $i = b$, before dealing with the other cases for values of $i$. If $i = 1$ we have many forced dominoes and we get the region shown in Figure 13, which is AD($a$). Again, if $i = b$, then we get a region of the type in Theorem 1.2 due to forced dominoes. In both of these cases the number of domino tilings of these regions satisfy the formula mentioned in the statement. From now on, we assume $b > 3$ and $1 < i < b$. We denote the region of the type shown in Figure 12 by $\mathcal{AR}^i_{a,b,k-1}$. We use Theorem 4.5 here, with the vertices $w, x, y$ and $z$ marked as shown in Figure 14, where we add a series of unit squares to the northeastern side to make it into an $a \times (b + 1)$ Aztec rectangle. Note that the square in the $i$-th position to be removed is included in this region and is labelled by $z$. The identity we now obtain is the following (see Figure 15 for forcings)

$$\text{M}(\text{AD}(a))\,\text{M}(\mathcal{AR}^i_{a,b+1,k}) = \text{M}(\text{AD}(a))\,\text{M}(\mathcal{AR}^i_{a,b,k-1}) + Y \cdot \text{M}(\mathcal{AR}_{a,b}(2, 3, \ldots, k, k+1)) \tag{4.7}$$

where

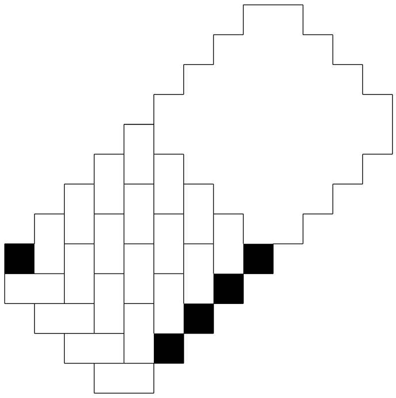


FIGURE 13. Forced tilings for $i = 1$ in Proposition 4.10.

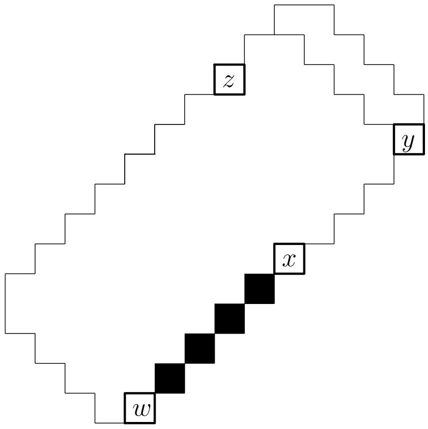


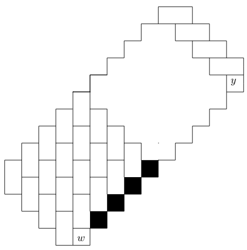

FIGURE 14. Labelled $a \times (b+1)$ Aztec rectangle; here $a = 4, b = 9$.

$$Y := \begin{cases} 0, & \text{if } i \leq k \\ \mathrm{M}(\mathcal{AR}_{a,a+1}(a+k+2-i)), & \text{if } i \geq k+1 \end{cases}. \tag{4.8}$$

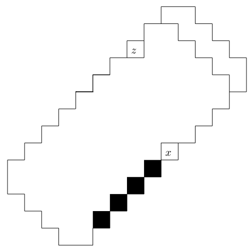
Using equation (4.8) in equation (4.7), we can simplify the relation further to the following

$$\mathrm{M}(\mathcal{AR}^i_{a,b+1,k}) = \mathrm{M}(\mathcal{AR}^i_{a,b,k-1}) + Z \cdot \mathrm{M}(\mathcal{AR}_{a,b}(2,3,\ldots,k+1)) \tag{4.9}$$

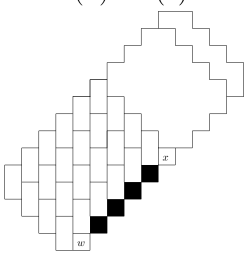
where



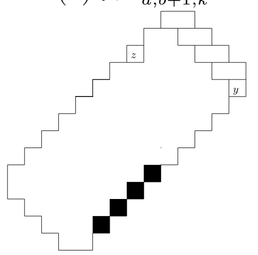

(A) AD($a$).



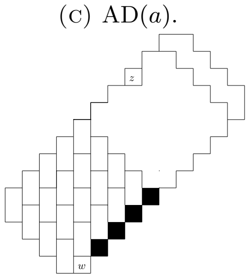

(B) $\mathcal{AR}^i_{a,b+1,k}$.



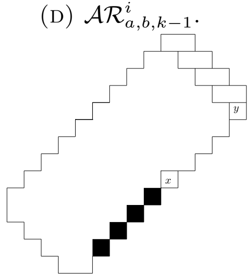

(C) AD($a$).




(D) $\mathcal{AR}^i_{a,b,k-1}$.




(E) $\mathcal{AR}_{a,a+1}(a+k+2-i)$.




(F) $\mathcal{AR}_{a,b}(2,3,\dots,k,k+1)$.

FIGURE 15. Forced dominoes in the proof of Proposition 4.9 where the vertices we remove are labelled

$$Z := \begin{cases} 0, & \text{if } i \le k \\ \dfrac{\mathrm{M}(\mathcal{AR}_{a,a+1}(a+k+2-i))}{\mathrm{M}(\mathrm{AD}(a))}, & \text{if } i \ge k+1 \end{cases}. \tag{4.10}$$

It now remains to show that the expression in the statement satisfies equation (4.9). This is now a straightforward application of the induction hypothesis and some algebraic manipulation (see the proof of Proposition 4.11 below for a more detailed proof). $\square$

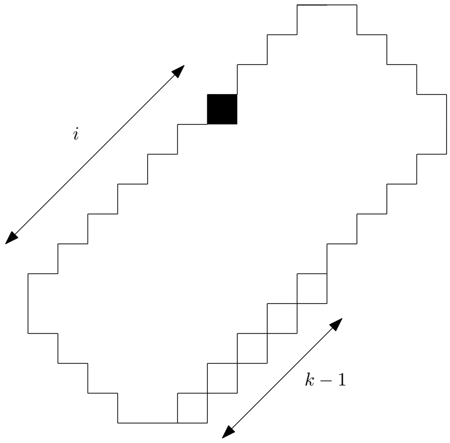



FIGURE 16. An Aztec rectangle with $k-1$ squares added on the southeastern side and a defect on the $i$-th position shaded in black; here $a = 4, b = 10, k = 6, i = 7$.

**Proposition 4.10.** *Let $1 \leq a, i \leq b$ be positive integers with $k = b - a > 0$ and $y = \min\{i, k\}$, then the number of domino tilings of $\mathcal{AR}_{a,b}$ with $k-1$ squares added to the southeastern side starting at the second position (and not at the bottom) as shown in the Figure 16 and a defect on the northwestern side at the $i$-th position counted from the western corner is given by*

$$2^{a(a+1)/2}\binom{a}{i-y}\sum_{l=0}^{y-1}\binom{a+y-l-2}{y-l-1}\,{}_3F_2\left[\begin{matrix}1, 1-y+l, i-a-y\\ i-y+1, 2-a-y+l\end{matrix}; -1\right]. \tag{4.11}$$

*Proof.* We follow a similar approach for this proof, like we did in the proof of Proposition 4.3. Let us denote the region in Figure 16 by $\mathcal{AR}_{a,b}(k-1; i)$ and we work with the planar dual graph of this region and count the number of matchings of that graph. We first notice that the first added square in any tiling of the region in Figure 16 by dominoes has two possibilities to be matched up with squares marked in grey in Figure 17. This observation allows us to write the number of tilings of $\mathcal{AR}_{a,b}(k-1; i)$ in terms of the following recursion (see Figure 18)

$$\mathrm{M}(\mathcal{AR}_{a,b}(k-1; i)) = \mathrm{M}(\mathcal{AR}_{a,b-1}(k-2; i-1)) + \mathrm{M}(\mathcal{AR}^{i}_{a,b,k-1}). \tag{4.12}$$

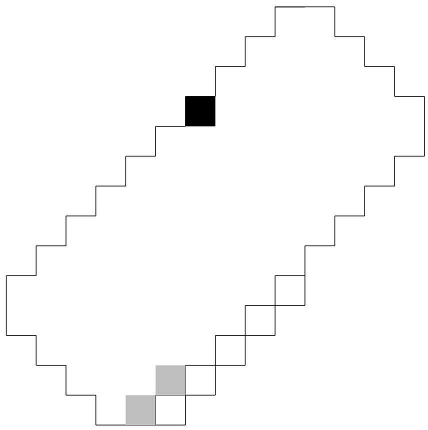


FIGURE 17. $\mathcal{AR}_{a,b}(k-1;i)$ with the possible choices for the first added square in a tiling; here $a=4, b=10, k=6, i=7$.

Figures/proof-new.pdf

FIGURE 18. Choices for the tilings of $\mathcal{AR}_{a,b}(k-1;i)$ with forced dominoes; here $a=4, b=10, k=6, i=7$.

As in the proof of Proposition 4.3, repeatedly using equation (4.12) $y-1$ times on successive iterations, we shall finally obtain

$$\mathrm{M}(\mathcal{AR}_{a,b}(k-1;i)) = \sum_{l=0}^{y-2} \mathrm{M}(\mathcal{AR}^{i-l}_{a,b-l,k-l}) + \mathrm{M}(\mathcal{AR}_{a,a+1}(i-y+1)) \tag{4.13}$$

where $y = \min\{i,k\}$.

Now, plugging in the values of the quantities in the right hand side of equation (4.13) from Proposition 4.9 and Corollary 4.1, and then transforming the binomial sum into hypergeometric notations we shall obtain equation (4.11).

□

**Proposition 4.11.** *Let $a, i, j$ be positive integers such that $1 \leq i, j \leq a$, then the number of domino tilings of* AD$(a)$ *with one defect on the southeastern side at the $i$-th position counted from the south corner and one defect on the northeastern side at the $j$-th position counted from the north corner as shown in Figure 19 is given by*

$$2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1}\,{}_3F_2\left[\begin{matrix}1, 1-i, 1-j\\ 1-a, 1-a\end{matrix}; 2\right].$$

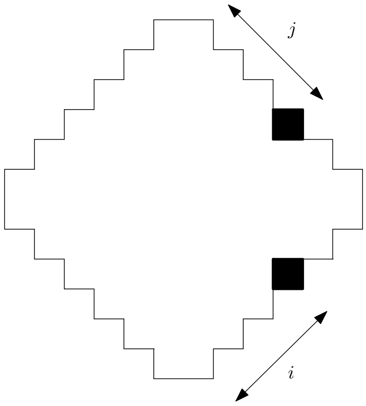



FIGURE 19. An Aztec diamond with defects on adjacent sides; here $a = 6$, $i = 4$, $j = 4$.

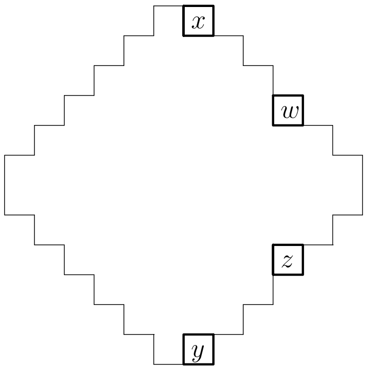



FIGURE 20. An Aztec diamond with some labelled squares; here $a = 6$.

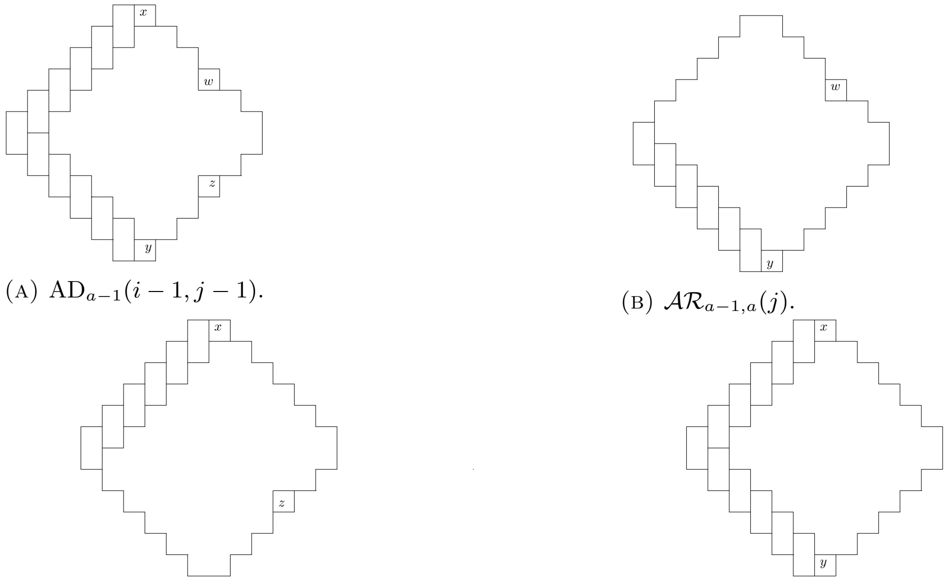
*Proof.* We use induction with respect to $a$. The base case of induction is $a = 2$, $i = 1, 2$, $j = 1, 2$. We check for the cases when, $i = 1, j = 1, i = a$ and $j = a$ separately. So, for our base case, the only possibilities are $i = 1$ or $i = a$ and $j = 1$ or $j = a$, so we do not have to consider this case, once we consider the other mentioned cases.

We now note that when either $i$ or $j$ is 1 or $a$, some dominoes are forced in any tiling and hence we are reduced to an Aztec rectangle of size $(a-1) \times a$. It is easy to see that our formula is correct for this.

In the rest of the proof we assume $a \geq 3$ and $1 < i, j < a$. Let us now denote the region we are interested in this proposition as $\mathrm{AD}_a(i, j)$. Using the dual graph of this region and applying Theorem 4.4 with the vertices as labelled in Figure 20 we obtain the following identity (see Figure 21 for details),

$$\begin{aligned} \mathrm{M}(\mathrm{AD}_a(i,j))\,\mathrm{M}(\mathrm{AD}(a-1)) = {} & \mathrm{M}(\mathrm{AD}(a))\,\mathrm{M}(\mathrm{AD}_{a-1}(i-1,j-1)) \\ & + \mathrm{M}(\mathcal{AR}_{a-1,a}(j))\,\mathrm{M}(\mathcal{AR}_{a-1,a}(i)). \end{aligned} \tag{4.14}$$

Simplifying equation (4.14), we get the following




(A) $\mathrm{AD}_{a-1}(i-1, j-1)$.

(B) $\mathcal{AR}_{a-1,a}(j)$.

(C) $\mathcal{AR}_{a,a+1}(i)$.

(D) $\mathrm{AD}(a-1)$.

FIGURE 21. Forced dominoes in the proof of Proposition 4.11 where the vertices we remove are labelled.

$$\mathrm{M}(\mathrm{AD}_a(i,j)) = 2^a\, \mathrm{M}(\mathrm{AD}_{a-1}(i-1, j-1)) + 2^{a(a-1)/2}\binom{a-1}{j-1}\binom{a-1}{i-1} \tag{4.15}$$

where we used Theorem 1.1 and Corollary 4.1.

Now, using our inductive hypothesis on equation (4.15) we have the following

$$\begin{aligned}
\mathrm{M}(\mathrm{AD}_a(i,j)) =& 2^a \cdot 2^{(a-1)(a-2)/2}\binom{a-2}{i-2}\binom{a-2}{j-2}\ {}_3F_2\left[\begin{matrix}1, 2-i, 2-j\\ 2-a, 2-a\end{matrix};2\right] \\
&+ 2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1} \\
=& 2 \cdot 2^{a(a-1)/2}\binom{a-2}{i-2}\binom{a-2}{j-2}\sum_{k=0}^{\infty}\frac{(1)_k(2-i)_k(2-j)_k}{(2-a)_k(2-a)_k}\frac{2^k}{k!} \\
&+ 2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1} \\
=& 2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1}\left[2\cdot\frac{(1-i)(1-j)}{(1-a)(1-a)}\sum_{k=0}^{\infty}2^k\cdot\frac{(2-i)_k(2-j)_k}{(2-a)_k(2-a)_k}+1\right] \\
=& 2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1}\sum_{k=0}^{\infty}2^k\cdot\frac{(1-i)_k(1-j)_k}{(1-a)_k(1-a)_k}.
\end{aligned}$$

This completes the proof of our proposition.

□

*Remark* 4.12. Ciucu and Fischer [2] have a similar result for the number of lozenge tilings of a hexagon with defects on adjacent sides (Proposition 3 in their paper). They also make use of Kuo's condensation result, Theorem 4.6 and obtain the following identity

$$\begin{aligned}\mathrm{ADJ}(a,b,c)_{j,k}\,\mathrm{ADJ}(a-1,b,c-1)_{j,k}\\ =\mathrm{ADJ}(a,b,c-1)_{j,k}\,\mathrm{ADJ}(a-1,b,c)_{j,k}\\ +\mathrm{ADJ}(a-1,b+1,c-1)_{j,k}\,\mathrm{ADJ}(a,b-1,c)_{j,k}\end{aligned}$$

where $\mathrm{ADJ}(a,b,c)_{j,k}$ denotes the number of lozenge tilings of a hexagon $H_{a,b,c}$ with opposite side lengths $a, b, c$ with two defects on adjacent sides of length $a$ and $c$ in positions $j$ and $k$ respectively, where $a, b, c, j, k$ are non-negative integers with $1 \leq j \leq a$ and $1 \leq k \leq c$.

In their use of Theorem 4.6, they take the graph $G$ to be the planar dual graph of $H_{a,b,c}$ with two defects on adjacent sides of length $a$ and $c$ in positions $j$ and $k$ (the resulting number of lozenge tilings of such a region is then $\mathrm{ADJ}(a,b,c)_{j,k}$), but if we take the graph $G$ to be the planar dual graph of $H_{a,b,c}$ and use Theorem 4.4 with an appropriate choice of labels we obtain the following identity

$$\begin{aligned}\mathrm{H}(a-1,b,c)\,\mathrm{ADJ}(a,b,c)_{j,k} &= \mathrm{H}(a,b,c)\,\mathrm{ADJ}(a-1,b,c)_{j,k}\\ &\quad+\mathrm{H}(c,a-1,b+1,c-1,a,b)_k\,\mathrm{H}(b-1,c+1,a-1,b,c,a)_j\end{aligned}$$

with the same notations as in Remark 4.8. Then, Proposition 3 of Ciucu and Fischer [2] follows more easily without the need for contiguous relations of hypergeometric series that they use in their paper.

## 5. Proofs of the main results

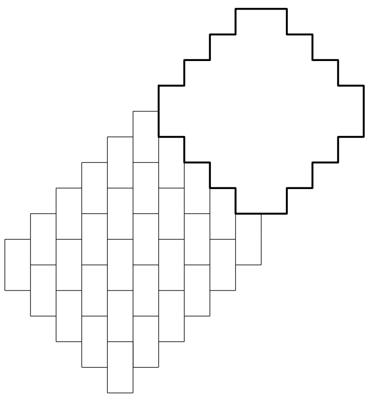


FIGURE 22. Removing the forced dominoes from $\mathcal{AR}_{a,b}^k$; here $a = 5, b = 10$, $k = 5$.

*Proof of Theorem 2.2.* We shall apply the formula in Theorem 3.1 to the planar dual graph of our region $\mathcal{AR}_{a,b}^k$, and the vertices $\delta_1, \ldots, \delta_{2n+2k}$. Then the left hand side of equation (3.2) becomes the left hand side of equation (2.1), and the right hand side of equation (3.2) becomes the right hand side of (2.1). We just need to verify that the quantities expressed in equation (2.1) are indeed given by the formulas described in the statement of Theorem 2.2.

The first statement follows immediately by noting that the added squares on the southeastern side of $\mathcal{AR}_{a,b}^k$ forces some dominoes. After removing this forced dominoes we are left with an Aztec diamond of order $a$ as shown in Figure 22, whose number of tilings is given by Theorem 1.1.

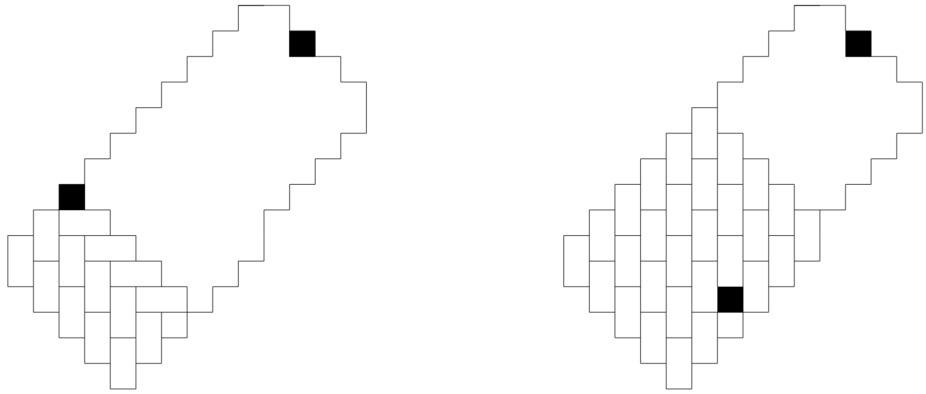


FIGURE 23. Choices of $\beta$ defects that lead to no tiling of $\mathcal{AR}_{a,b}^k$.

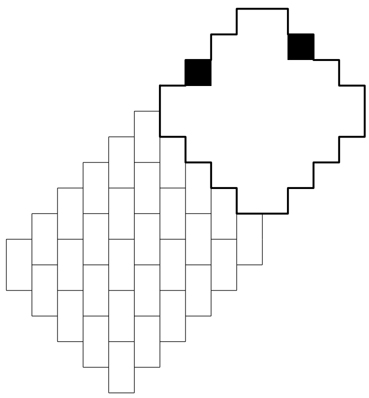


FIGURE 24. Choice of $\beta$-defect, not sharing an edge with some $\gamma_l$.

The possibilities in the second statement are as follows. If an $\beta$ square shares an edge with some $\gamma_l$, then the region cannot be covered by any domino as illustrated in the right image of

Figure 23. Again, if $\beta_i$ is on the northwestern side at a distance of at most $k$ from the western corner, then the strips of forced dominoes along the southwestern side interfere with the $\beta_i$ and hence there cannot be any tiling in this case as illustrated in the left image of Figure 23. If neither of these situations is the case, then due to the squares $\gamma_1, \ldots, \gamma_k$ on the southeastern side, there are forced dominoes as shown in Figure 24 and then $\beta_i$ and $\alpha_j$ are defects on an Aztec diamond on adjacent sides and then the second statement follows from Proposition 4.11.

Figures/al-ga-no-no-opp.pdf

FIGURE 25. Choices of $\beta$ and $\gamma$-defects that lead to no tiling of $\mathcal{AR}_{a,b}^k$.

To prove the validity of the third statement, we notice that if an $\beta$ and $\gamma$ defect share an edge then, there are two possibilities, either the $\beta$ defect is above the $\gamma$ defect in which case we have some forced dominoes as shown in the left of Figure 26 and we are reduced to finding the number of domino tilings of an Aztec diamond; or the $\beta$-defect is to the left of a $\gamma$ dent, in which case, we get no tilings as shown in the left of Figure 25 as the forced dominoes interfere in this case.

If $\beta_i$ and $\gamma_j$ share no edge in common, then we get no tiling if the $\beta$-defect is on the northwestern side at a distance of at most $k-1$ from the western corner as illustrated in the right of Figure 25. If the $\beta$-defect is in the northwestern side at a distance more than $k-1$ from the western corner then the situation is as shown in the right of Figure 26 and is described in Proposition 4.10. If the $\beta$-defect is in the southeastern side then the situation is as shown in the middle of Figure 26 and is described in Proposition 4.3.

Figures/proof-new-last.pdf

FIGURE 26. Choices of $\beta$ and $\gamma$-defects that lead to tiling of $\mathcal{AR}_{a,b}^k$.

The fourth statement follows immediately from the checkerboard drawing (see Figure 2) of an Aztec rectangle and the condition that a tiling by dominoes exists for such a board if and only if the number of white and black squares are the same. In all other cases, the numbers of tilings is 0.

□

*Proof of Theorem 2.3.* Let $\mathcal{AR}$ be the region obtained from $\mathcal{AR}_{a,b}^k$ by removing $k$ of the squares $\beta_1, \ldots, \beta_{n+k}$. We now apply Theorem 3.1 to the planar dual graph of $\mathcal{AR}$, with the removed squares chosen to be the vertices corresponding to the $n$ $\beta_i$'s inside $\mathcal{AR}$ and to $\alpha_1, \ldots, \alpha_n$. The left hand side of equation (3.2) is now the required number of tilings and the right hand side of equation (3.2) is the Pfaffian of a $2n \times 2n$ matrix with entries of the form $\mathrm{M}(\mathcal{AR} \setminus \{\beta_i, \alpha_j\})$, where $\beta_i$ is not one of the unit squares that we removed from $\mathcal{AR}_{a,b}^k$ to get $\mathcal{AR}$.

We now notice that $\mathrm{M}(\mathcal{AR} \setminus \{\beta_i, \alpha_j\})$ is an Aztec rectangle with all its defects confined to three of the sides. So, we can apply Theorem 2.2 and it gives us an expression for $\mathrm{M}(\mathcal{AR} \setminus \{\beta_i, \alpha_j\})$

as the Pfaffian of a $(2k+2)\times(2k+2)$ matrix of the type described in the statement of Theorem 2.2. □

*Proof of Theorem 2.4.* We shall now apply Theorem 3.1 to the planar dual graph of $\mathrm{AD}(a)$ with removed squares chosen to correspond to $\beta_1,\dots,\beta_n,\alpha_1,\dots,\alpha_n$. The right hand side of equation (3.2) is precisely the right hand side of equation (2.2). If $\delta_i$ and $\delta_j$ are of the same type then $\mathrm{AD}(a)\setminus\{\delta_i,\delta_j\}$ does not have any tiling as the numbers of black and white squares in the checkerboard setting of an Aztec diamond will not be the same (see Figure 2). Finally, the proof is complete once we note that $\mathrm{AD}(a)\setminus\{\beta_i,\alpha_j\}$ is an Aztec diamond with two defects removed from adjacent sides for any choice of $\beta_i$ and $\alpha_j$ and is given by Proposition 4.11. □

We now illustrate Theorem 2.2 with the help of an example.

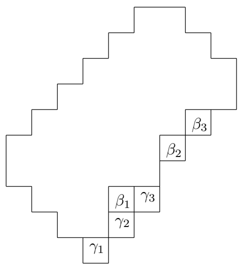



FIGURE 27. $\mathcal{AR}^3_{3,6}$ with the $\alpha,\beta,\gamma$ defects marked.

**Example 5.1.** Throughout this example, we will use the notations from previous sections without commentary. We consider the example in Figure 27 with $a=3, b=6, k=3, n=0$ and the set $\{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6\}=\{\gamma_1,\gamma_2,\beta_1,\gamma_3,\beta_2,\beta_3\}$. Then by equation (2.1) we have

$$\mathrm{M}(\mathcal{AR}_{3,6}\setminus\{\beta_1,\beta_2,\beta_3\})=\frac{1}{[\mathrm{M}(\mathrm{AD}(3))]^2}\,\mathrm{Pf}(A)$$

where the matrix $A$ is

$$\begin{pmatrix}
0 & 0 & \mathrm{M}(\mathrm{AD}(3)) & 0 & \mathrm{M}(\mathcal{AR}^{2,4}_{3,6}) & \mathrm{M}(\mathcal{AR}^{2,5}_{3,6})\\
0 & 0 & \mathrm{M}(\mathrm{AD}(3)) & 0 & \mathrm{M}(\mathcal{AR}^{1,3}_{3,5}) & \mathrm{M}(\mathcal{AR}^{1,4}_{3,5})\\
-\mathrm{M}(\mathrm{AD}(3)) & -\mathrm{M}(\mathrm{AD}(3)) & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \mathrm{M}(\mathcal{AR}_{3,4}(2)) & \mathrm{M}(\mathcal{AR}_{3,4}(3))\\
-\mathrm{M}(\mathcal{AR}^{2,4}_{3,6}) & -\mathrm{M}(\mathcal{AR}^{1,3}_{3,5}) & 0 & -\mathrm{M}(\mathcal{AR}_{3,4}(2)) & 0 & 0\\
-\mathrm{M}(\mathcal{AR}^{2,5}_{3,6}) & -\mathrm{M}(\mathcal{AR}^{2,4}_{3,5}) & 0 & -\mathrm{M}(\mathcal{AR}_{2,3}(3)) & 0 & 0
\end{pmatrix}.$$

If we now calculate all the quantities that appear in the matrix $A$ using the results mentioned in this paper, we shall get $\mathrm{Pf}(A)=3932160$ and hence

$$\mathrm{M}(\mathcal{AR}_{3,6}\setminus\{\beta_1,\beta_2,\beta_3\})=960.$$

It is easy to see that this also agrees if we use Theorem 1.2 to calculate $\mathrm{M}(\mathcal{AR}_{3,6}\setminus\{\beta_1,\beta_2,\beta_3\})$.

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