Alexandre Laugier, Manjil P. Saikia

A COMBINATORIAL PROOF OF A RESULT ON GENERALIZED LUCAS POLYNOMIALS

Abstract. We give a combinatorial proof of an elementary property of generalized Lucas Polynomials, inspired by [1]. These polynomials in s and t are defined by the recurrence relation $\langle n\rangle = s\langle n-1\rangle + t\langle n-2\rangle$ for $n \geq 2$. The initial values are $\langle 0\rangle = 2, \langle 1\rangle = s$ respectively.

1. Introduction and Motivation

In this paper, we shall focus on giving a combinatorial proof of a result on the generalized Lucas polynomials. But first we give some introductory remarks and motivation. The famous *Fibonacci numbers*, F_n are defined by $F_0 = 0$, $F_1 = 1$ and, for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

The *Lucas numbers* L_n are defined by the same recurrence, with the initial conditions $L_0 = 2$ and $L_1 = 1$.

One generalization of these numbers which has received much attention is the sequence of *Fibonacci polynomials*

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 2,$$

with initial conditions $F_0(x) = 0$, $F_1(x) = 1$. The *generalized Fibonacci polynomials* depend on two variables s, t and are defined by $\{0\}_{s,t} = 0$,

2010 *Mathematics Subject Classification*: Primary 05A10; Secondary 11B39, 11B65.

Key words and phrases: binomial theorem, Fibonacci number, Fibonomial coefficient, Lucas number q-analogue, Generalized Lucas Polynomials.

$\{1\}_{s,t} = 1$ and, for $n \geq 2$,

$$\{n\}_{s,t} = s\{n-1\}_{s,t} + t\{n-2\}_{s,t}.$$

Here and with other quantities depending on s and t, we will drop the subscripts as they will be clear from context. For example, we have

$$\{2\} = s, \quad \{3\} = s^2 + t, \quad \{4\} = s^3 + 2st, \quad \{5\} = s^4 + 3s^2t + t^2.$$

For some historical remarks and relations of these polynomials we refer the reader to [1], [2] and [3].

The main focus of our paper are the generalized Lucas polynomials defined by

$$\langle n \rangle_{s,t} = s\langle n-1 \rangle_{s,t} + t\langle n-2 \rangle_{s,t}, \quad n \geq 2$$

together with the initial conditions $\langle 0 \rangle_{s,t} = 2$ and $\langle 1 \rangle_{s,t} = s$. The first few polynomials are

$$\langle 2 \rangle_{s,t} = s^2+2t, \quad \langle 3 \rangle_{s,t} = s^3+3st, \quad \langle 4 \rangle_{s,t} = s^4+4s^2t+2t^2, \quad \langle 5 \rangle_{s,t} = s^5+5s^3t+5st^2.$$

When $s = t = 1$ these reduce to the ordinary Lucas numbers.

2. Combinatorial Interpretations of $\{n\}$ and $\langle n \rangle$

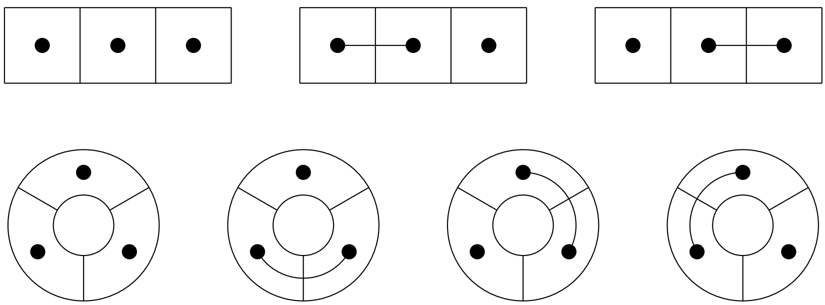

Figure 1: Linear and circular tilings

In addition to the algebraic approach to our polynomials, there is a combinatorial interpretation derived from the standard interpretation of F_n via tiling, given in [3]. A *linear tiling*, T, of a row of squares is a covering of the squares with dominos (which cover two squares) and monominos (which cover one square). We let,

$$\mathcal{L}_n = \{T \ : \ T \text{ a linear tiling of a row of } n \text{ squares}\}.$$

The three tilings in the first row of Figure 1 are the elements of $\mathcal{L}_3$. We will also consider *circular tilings* where the (deformed) squares are arranged in a circle. We will use the notation $\mathcal{C}_n$ for the set of circular tilings of n squares. So the set of tilings in the bottom row of Figure 1 is $\mathcal{C}_3$. For any type of nonempty tiling, T, we define its *weight* to be

$$\operatorname{wt} T = s^{\#\text{ of monominos in } T} t^{\#\text{ of dominos in } T}.$$

We give the empty tiling ϵ of zero boxes the weight $\operatorname{wt}\epsilon = 1$, if it is being considered as a linear tiling or $\operatorname{wt}\epsilon = 2$, if it is being considered as a circular tiling. The following proposition is immediate from the definitions of weight and of our generalized polynomials.

Proposition 2.1 (Sagan and Savage, [3]). *For $n \geq 0$, we have*

$$\{n+1\} = \sum_{T \in \mathcal{L}_n} \operatorname{wt} T$$

and

$$\langle n \rangle = \sum_{T \in \mathcal{C}_n} \operatorname{wt} T.$$

From the above discussions on the combinatorial interpretations of $\{n\}$ and $\langle n \rangle$ we get the following.

Theorem 2.2 (Sagan and Savage, [3]). *For $m \geq 1$ and $n \geq 0$ we have*

$$\{m+n\} = \{m\}\{n+1\} + t\{m-1\}\{n\}.$$

Proposition 2.3 (Sagan and Savage, [3]). *For $n \geq 1$ we have*

$$\langle n \rangle = \{n+1\} + t\{n-1\}.$$

And for $m, n \geq 0$ we have

$$\{m+n\} = \frac{\langle m \rangle \{n\} + \{m\} \langle n \rangle}{2}.$$

For some more interesting combinatorial interpretations, we refer the reader to [1] and [3].

3. Main Result

The main aim of this paper is to give a combinatorial proof of the following result, inspired by [1].

Theorem 3.1. *For* $s, t \in \mathbb{N}$ *such that* $\frac{1}{s+t} < \min\left\{\frac{1}{|X|}, \frac{1}{|Y|}\right\}$*, we have for* $X, Y \neq 0$

$$\sum_{n=0}^{\infty} \frac{\langle n \rangle_{s,t}}{(s+t)^{n+1}} = \frac{s+2t}{t(s+t-1)}.$$

Proof. We consider an infinite row of squares which extends to both directions. A random square is marked as the 0^{th} place. The squares are numbered from left to right of 0 by the positive integers and from the right to left of 0 by the negative integers.

We now suppose that each square can be coloured with one of s shades of white and t shades of black. Let Z be the random variable which returns the box number at the end of the first odd-length block of boxes of the same black shade starting from the right of 0, and let Z' be the random variable which returns the box number at the end of the first odd-length block of boxes of the same black shade from the left of 0. And let W be the event be the combination of both Z and Z'.

For any integer n, the event $W = n$ is the combination of $Z = n$ and $Z' = -n$. Here $Z = n$ is equivalent to having box n painted with one of the shades of black among the first n squares being of even length including 0 to the right of 0. So there are t choices for the colour of box n and $s + t - 1$ choices for the colour of box $n + 1$. Similarly $Z' = -n$ is equivalent to having box $-n$ painted with one of the shades of black among the first n squares being of even length including 0 to the left of 0. So, there are t choices for the colour of box $-n$ and $s + t - 1$ choices for the colour of box $-(n + 1)$.

Each colouring of the first n squares gives a tiling where each white box is replaced by a monomino and a block of $2k$ boxes of the same shade of black is replaced by k dominoes. Also, the weight of the tiling is just the number of colourings attached to it. Thus, the number of the colourings of the first n boxes is $\langle n \rangle$ since the n boxes both to the right and left of 0 will give rise to a circular tiling in this case. Indeed, the number of the colourings of the first n boxes to the right of 0, including the box 0 is $\{n + 1\}$. Moreover, if the number of black shades boxes to the left of 0, not including the box 0, is even, then the number of the colourings of the first $n - 1$ boxes to the left of 0, not including the box 0 is $\{n\}$. By convention, we fix the shade of the box

0 to be white. Since there are s possible white shades for the box 0, there are $s\{n\}$ colourings of the first n boxes to the left of 0, including the box 0. So, there are $\{n+1\} - s\{n\} = t\{n-1\}$ colourings of the first $n-1$ boxes to the left of 0, not including the box 0. This implies that the number of the colourings of the first n boxes is $\{n+1\} + t\{n-1\} = \langle n \rangle$.

Notice that if the shade of the box 0 is white, then the box 0 contributes by a factor s to the total number of circular tilings for which $W = n$ whereas if the shade of the box 0 is black, then the box 0 contributes by a factor $2t$ to the total number of circular tilings since for each black shade, there are two possibilities (namely the two neighbours of box 0 in a circular tiling). It gives rise to a multiplicative factor $s + 2t$ in the expression of the total number of circular tilings for which $W = n$. Notice that once we count $s + 2t$ for the box 0, the other boxes (including the box $n+1$) contributes by a multiplicative factor $s + t$ to the total number of circular tiling. Thus, the total number of circular tilings for which $W = n$ is given by $(s+2t)(s+t)^{n+1}$.

Hence we have,

$$P(W = n) = \frac{t(s+t-1)\langle n \rangle_{s,t}}{(s+2t)(s+t)^{n+1}}.$$

Summing these will give us the desired result. ∎

Acknowledgements

The second author was supported by DST INSPIRE Scholarship 422/2009 from the Department of Science and Technology, Government of India when this work was done.

References

[1] T. Amdeberhan, X. Chen, V. H. Moll, B. E. Sagan, *Generalized Fibonacci polynomials and Fibonomial coefficients*, Ann. Comb., **18** (4) (2014), 541 – 562.

[2] S. Ekhad, *The Sagan-Savage Lucas-Catalan polynomials have positive coefficients*, preprint `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/bruce.html`

[3] B. E. Sagan, C. D. Savage, *Combinatorial interpretations of binomial coefficient analogues related to Lucas sequences*, Integers, A52, **10** (2010), 697 – 703.

A. Laugier
LYCÉE TRISTAN CORBIÈRE
16 RUE DE KERVÉGUEN - BP 17149 - 29671
MORLAIX CEDEX, FRANCE
E-mail: laugier.alexandre@orange.fr

M. P. Saikia
DIPLOMA STUDENT, MATHEMATICS GROUP
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
11, STRADA COSTIERA
TRIESTE, ITALY
E-mail: msaikia@ictp.it, manjil@gonitsora.com