PÓLYA'S THEORY OF COUNTING

MANJIL P. SAIKIA

Abstract. In this report, we shall study Pólya's Theory of Counting and apply it to solve some questions related to rotations of the platonic solids.

Key Words: Pólya's Theory of Counting, group action, platonic solids.

2010 Mathematical Reviews Classification Numbers: 05A99, 05E18.

1. Introduction

In 1937, George Pólya wrote what is considered to be one of the most significant papers in combinatorics. The theorem that he introduced is now studied as Pólya's Theory of Counting and has wide range of applications not only in mathematics but also in other branches of science, most notably in chemistry. This report is divided into three sections, the first section are some brief introductory remarks and notations. The second section discusses Pólya's theory in details and the final section, uses Pólya's Theory to find out the rotations of the platonic solids.

Let A and B be finite sets and G be a group of permutations of A. The group G acts on the set B^A of mappings $f : A \to B$ when for $\sigma \in G$ and $f \in B^A$, we define $\sigma(f) \in B^A$ by

$$(\sigma(f))(x) := f(\sigma^{-1}(x)).$$

It is the orbits of an appropriate G on the set B^A that we would be counting mainly.

2. Pólya's Theory of Counting

In this section, we closely follow the excellent treatment given in [3]. Before proceeding to the main theorems, we recall the following classical result from group theory without proof.

Lemma 2.1 (Burnside)**.** *The number of orbits of a finite group G on a set X is the average number of fixed points*

$$\frac{1}{|G|}\sum_{\sigma\in G}\psi(\sigma),$$

where $\psi(\sigma)$ denotes the number of points of S that are fixed by the permutation σ.

We now start with the following theorem.

Theorem 2.2. *Let A and B be finite sets and let G act on A. We denote by $c_k(G)$ the number of permutations of G that have exactly k cycles in their decomposition on A. Then the number of orbits of G on the set B^A of all mappings $f : A \to B$ is*

$$\frac{1}{|G|}\sum_{k=1}^{\infty} c_k(G)|B|^k.$$

Proof. By Burnside's Lemma, the number of orbits is given by

$$\frac{1}{|G|}\sum_{\sigma\in G}\psi(\sigma),$$

where $\psi(\sigma)$ denotes the number of mappings $f : A \to B$ such that $\sigma(f) = f$, that is $f(a) = f(\sigma^{-1}(a))$ for all $a \in A$.

But a mapping f is fixed by σ if and only if f is constant on every cycle of σ. Such mappings are obtained by assigning an element of B to each cycle of σ, and thus if σ has k cycles, the number of mappings f fixed by σ is $|B|^k$ from which the result follows.

□

We shall use some notations in the remainder of this section that we now state here. The notation $(1^{k_1}2^{k_2}\dots n^{k_n})$ is used for the partition of the integer n which has k_i parts of size i, $i = 1, 2, \dots, n$. For a permutation σ of A, we denote the number of cycles of σ having length i as $z_i(\sigma)$. Given a group G acting on A, we define the cycle index Z_G as a polynomial in n letters $X_1, X_2, \dots, X_n$ by

$$Z_G(X_1, X_2, \dots, X_n) := \frac{1}{|G|}\sum_{\sigma\in G} X_1^{z_1(\sigma)}\cdots X_n^{z_n(\sigma)}.$$

Example 2.3. *A cyclic group C_n of order n has $\phi(d)$ elements of order d for each divisor d of n. As a permutation in the regular representation of C_n, an element of order d has n/d cycles of length d. Thus for C_n we have*

$$Z_{C_n}(X_1, \dots, X_n) = \frac{1}{n}\sum_{d|n}\phi(d)X_d^{n/d}.$$

Example 2.4. *For the dihedral group we have the following*

$$Z_{D_n} = \begin{cases} \frac{1}{2n}(\sum_{d|n}\phi(d)X_d^{n/d} + nX_1X_2^{\frac{n-1}{2}}) & \textit{for odd } n \\ \frac{1}{2n}(\sum_{d|n}\phi(d)X_d^{n/d} + \frac{n}{2}X_1^2X_2^{\frac{n}{2}-1} + \frac{n}{2}X_2^{\frac{n}{2}}) & \textit{otherwise.} \end{cases}$$

We shall now generalize Theorem 2.2 to allow for weights. Let A be an n-set, let G act on A, and let B be a finite set of colours. Let R be a commutative ring containing the rationals and let $w : B \to R$ assign a weight $w(b) \in R$ to each colour $b \in B$. For $f : A \to B$, we define

$$W(f) := \prod_{a\in A} w(f(a)) \in R;$$

here $W(f)$ is called the weight of the function f. We note that two mappings representing the same orbit of G on B^A have the same weight, that is $W(\sigma(f)) = W(f)$ for every $\sigma \in G$. The sum

$$\sum_{f\in R} W(f),$$

extended over a system of representatives R for the orbits is called the *configuration counting series*.

Theorem 2.5. *With the terminology as above, the configuration counting series is given by*

$$\sum W(f) = Z_G\left(\sum_{b\in B} w(b), \sum_{b\in B}[w(b)]^2, \dots, \sum_{b\in B}[w(b)]^n\right).$$

If all weights $w(b)$ are 1, then the theorem reduces to Theorem 2.2.

Proof. Let $N = \sum W(f)$, the sum extended over all pairs (σ, f) with $\sigma \in G$, $f \in B^A$, and $\sigma(f) = f$. We have

$$N = \sum_{f \in B^A} W(f)|G_f|,$$

where G_f is the stabilizer of f. We now consifder the terms $W(f)|G_f|$ as f ranges over an orbit $\mathcal{O}$ of G on B^A. each of which is equal to $W(f_0)|G|/|\mathcal{O}|$ where f_0 is a representative of the orbit $\mathcal{O}$, so these sums to $|G|W(f_0)$. Thus, it is clear that N is $|G|$ times the configuration counting series.

Again, we have

$$N = \sum_{\sigma \in G} \left(\sum_{\sigma(f)=f} W(f) \right).$$

From the definition of Z_G it is clear that we need to show

$$\sum_{\sigma(f)=f} W(f) = \left(\sum_{b \in B} w(b) \right)^{k_1} \left(\sum_{b \in B} [w(b)]^2 \right)^{k_2} \cdots \left(\sum_{b \in B} [w(b)]^n \right)^{k_n}$$

whenever σ is a permutation of type $(1^{k_1} 2^{k_2} \cdots n^{k_n})$.

A mapping $f : A \to B$ is fixed by σ if and only if f is constant on every cycle of σ on A. Let $C_1, C_2, \ldots, C_k$ be the cycles of σ where $k = k_1 + k_2 + \cdots + k_n$. The mappings $f \in B^A$ fixed by σ are in one-to-one correspondence with k-tuples $(b_1, b_2, \ldots, b_k)$ of elements of B and the corresponding mapping is the one associating b_i to all elements of C_i. The weight of the mapping f corresponding to $(b_1, \ldots, b_k)$ is given by

$$W(f) = \prod_{i=1}^{k} [w(b_i)]^{|C_i|},$$

and summing over all k-tuples $(b_1, b_2, \ldots, b_k)$ we get

$$\begin{aligned} \sum_{\sigma(f)=f} W(f) &= \sum_{b_1, b_2, \ldots, b_n} [w(b_1)]^{|C_1|} [w(b_2)]^{|C_2|} \cdots [w(b_k)]^{|C_k|} \\ &= \left(\sum_{b_1} [w(b_1)]^{|C_1|} \right) \left(\sum_{b_2} [w(b_2)]^{|C_2|} \right) \cdots \\ &= \prod_{c=1}^{n} \left(\sum_{b} [w(b)]^c \right)^{k_c}. \end{aligned}$$

□

3. Applications to Platonic Solids

We shall now use the results described above and try to find some of the symmetries of the platonic solids: tetrahedron, cube, octahedron, dodecahedron and icosahedron. Since the cube is the dual of the octahedron and the dodecahedron is the dual of the icosahedron, so we shall only study the tetrahedron, cube and the dodecahedron here. We follow the treatment given in [1] and [2].

3.1. **Tetrahedron.** We shall consider the group G of all rotations of a regular tetrahedron. The identity permutation has cycle type $(4, 0)$ here and the associated monomial is X_1^4. The non-identity rotations in this case are of two types: vertex based and edge based.

For the vertex based rotations, we fix a vertex and the center of the antipodal face taking the straight line joining them as the axis. We can have two rotations of 120^0 each, giving us two three cycles respectively. If we fix the vertex numbered 1 say, then the two three cycles will be (234) and (243). We have $4 \times 2 = 8$ such rotations of this kind and the associated monomial of each such rotation is X_1X_3.

For the next type of rotations, we consider two antipodal edges such as (14) and (23) and take an axis that joins the mid-points of these edges. We can rotate through 180^0 along this axis. Each such rotation will produce 2-cycles and the associated monomial is X_2^2. There are three such rotations.

We get a total of 12 rotations of the regular tetrahedron and thus

$$Z_G = \frac{1}{12}(X_1^4 + 8X_1X_3 + 3X_2^2).$$

The symmetry group, G is isomorphic to A_4.

3.2. **Cube.** Let G denote the rotation group of a cube C in its action on the six faces of the cube. Let F, B, L, R, U, D denote the front, back, left wall, right wall, top (up) and bottom (down) faces of C respectively. The identity here has cycle type $(6, 0)$ and the associated monomial is X_1^6. Besides the identity we have the following different types of rotations.

(1) *Vertex based rotations*: We fix an axis passing through a pair of antipodal vertices such as the meeting point of F, L and D the meeting point of B, R and U. We can have two rotations of 120^0 in this setting. This gives us three rotations with the given axis and they have the cycle type $(0, 0, 2)$ with associated monomial X_3^2. Since there are 4 pairs of antipodal vertices, we get $4 \times 2 = 8$ such rotations.

(2) *Edge based rotations*: We fix an axis passing through a pair of antipodal edges with an axis joining the mid-points of these edges such as the edge common to F and D and the edge common to B and U. We can rotate through an angle of 180^0 around this axis. This gives us a cylce type of $(0, 3)$ with the associated monomial X_2^3. The number of such rotations is 6 as we have 6 pairs of such antipodal edges.

(3) *Face based rotations*: We fix an axis passing through the centres of a pair of two opposite faces such as U and D. With the line joining these two points as the axis we can rotate C through an angle of 90^0. This gives us four rotations of which one is the identity and the others are $(FRBL)(U)(D)$ and its inverse both with the cycle type $(2, 0, 0, 1)$ and the square of this rotation (for 180^0 rotation) which has the cycle decomposition $(FB)(RL)$ with the cycle type $(2, 2, 0)$. The axis can be choosen in three ways and we have 6 monomials each equal to $X_1^2X_4$ corresponding to a 90^0 rotation and 3 monomials equal to $X_1^2X_2^3$ corresponding to a 180^0 rotation.

We get a total of 24 rotations of the cube and thus

$$Z_G = \frac{1}{24}(X_1^6 + 8X_3^2 + 6X_2^3 + 6X_1^2X_4 + 3X_1^2X_2^2).$$

The symmetry group, G is isomorphic to S_4.

3.3. **Dodecahedron.** We now consider a regular dodecahedron D with G, the group of all the rotations of D in its action on the 12 faces of D. Here the identity will have a cylce type $(12, 0, 0)$ while the other rotations can be classified as below.

(1) *Vertex based rotations*: We fix a pair of antipodal vertices with an axis passing through these two vertices. We rotate D through an angle of 120^0 and 240^0 around this axis. Since each vertex is incident with three faces, the cycle type of such a rotation is $(0,0,4)$ and the associated monomial is X_3^4. We have 10 pairs of antipodal vertices and hence the number of such rotations is equal to $10 \times 2 = 20$.

(2) *Edge based rotations*: We take a pair of antipodal edges and fix the mid-point of these edges. With an axis through these two mid-points, we rotate D through an angle of 180^0. This rotation pairs the 12 faces into 6 pairs and hence has cycle type $0,6)$ with the associated monomial being X_2^6. Since D has 30 edges, so we have 15 such rotations.

(3) *Face based rotations*: We take a pair of opposite faces and fix the centers of these faces. With an axis through the centers of these faces, we rotate D through an angle of 72^0 and then take multiples of 72^0. This creates 4 proper rotations, each such rotation fixes two faces and rotates the other ten in two 5-cycles (since each face is surrounded by 5 faces) giving the cycle type $(2,0,0,0,2)$. Since we have 12 faces and hence 6 pairs of opposite faces, we get $6 \times 4 = 24$ such rotations with the associated monomial being $X_1^2 X_5^2$.

We get a total of 60 rotations of the cube and thus

$$Z_G = \frac{1}{60}(X_1^{12} + 20X_3^4 + 15X_2^6 + 24X_1^2X_5^2).$$

The symmetry group, G is isomorphic to A_5.

ACKNOWLEDGEMENTS

I would like to thank Prof. Fernando R. Villegas for his guidance during the period of study.

REFERENCES

[1] M. Fantahun, *Pólya's Theory of Counting*, Diploma Thesis, The Abdus Salam International Centre for Theoretical Physics, August 2013, 43 pp.

[2] S. S. Sane, *Combinatorial Techniques*, Texts and Readings in Mathematics **64**, Hindustan Book Agency, New Delhi, 2013.

[3] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, 2 ed., Cambridge University Press, Cambridge, 2001.

DIPLOMA STUDENT, MATHEMATICS GROUP, THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS, STRADA COSTIERA-11, I-34151, MIRAMARE, TRIESTE, ITALY

E-mail address: manjil@gonitsora.com, msaikia@ictp.it