Solution to Problem 1908

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Problem 0.1. *Let ABC be an arbitary triangle with $a = BC, b = AC$ and $c = AB$. Denote by s, r and R, the semiperimeter, inradius and the circumradius of $\triangle ABC$ respectively. Prove that if m and n are positive real numbers, then the following inequalities hold:*

(a) $\frac{a}{mb+nc} + \frac{b}{mc+na} + \frac{c}{ma+nb} \geq \frac{4s^2}{(m+n)(s^2+r^2+4Rr)}$

(b) $\frac{1}{(ma+nb)^2} + \frac{1}{(mb+nc)^2} + \frac{1}{(mc+na)^2} \geq \frac{27}{4(m+n)^2 s^2}$.

Solution 0.1. (a) *We shall use the following facts in our solution*

$$\Delta^2 = (s-a)(s-b)(s-c),$$

$$r = \frac{\Delta}{s},$$

and

$$R = \frac{abc}{\Delta},$$

where Δ denotes the area of $\triangle ABC$.

We have

$$\begin{aligned} r^2 + s^2 + 4Rr &= \frac{\Delta^2}{s^2} + s^2 + \frac{4abc\Delta}{4\Delta s} \\ &= \frac{\Delta^2}{s^2} + s^2 + \frac{abc}{s} \\ &= \frac{(s-a)(s-b)(s-c)}{s} + s^2 + \frac{abc}{s} \\ &= \frac{(s-a)(s-b)(s-c) + s^3 + abc}{s} \\ &= ab + bc + ca. \end{aligned} \tag{0.1}$$

The last step in the above follows after simplying the expressions in the numerator.

Now by the Cauchy-Schwarz inequality we have

$$\left(\frac{a}{mb+nc} + \frac{b}{mc+na} + \frac{c}{ma+nb}\right)(a(mb+nc)+b(mc+na)+c(ma+nb)) \geq (a+b+c)^2.$$

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Putting $a+b+c=2s$ *and* $a(mb+nc)+b(mc+na)+c(ma+nb)=(m+n)(ab+bc+ca)$ *and using* (0.1) *in the above inequality we get the desired inequality. Here equality occurs when the triangle is equilateral.*

(b) *Let us denote* $(ma+nb)$, $(mb+nc)$ *and* $(mc+na)$ *by* x, y *and* z *respectively. We know from above that* $4(m+n)^2s^2=((ma+nb)+(mb+nc)+(mc+na))^2$. *Then the desired inequality is given by*

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geq\frac{27}{(x+y+z)^2}. \tag{0.2}$$

From the AM-GM inequality we have the following for positive x, y *and* z,

$$(x^2y^2+y^2z^2+z^2x^2)(x+y+z)^2\geq 27x^2y^2z^2.$$

The above is nothing but (0.2). *Now substituting the values of* x, y *and* z *in* (0.2) *we get the desired inequality. Here equality occurs when* $\triangle ABC$ *is equilateral.*

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