Enumeration of Domino Tilings of an Aztec Rectangle with boundary defects

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Table of contents

Introduction and Motivation

Aztec Diamonds

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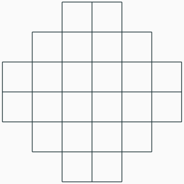

Figure 1: AD(3), Aztec Diamond of order 3

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Theorem (Elkies–Kuperberg–Larsen–Propp)
The number of domino tilings of an Aztec Diamond of order n is $2^{n(n+1)/2}$.

Aztec Rectangles

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- The non-tileability of the region $\mathcal{AR}_{a,b}$ becomes evident if we look at the checkerboard representation of $\mathcal{AR}_{a,b}$

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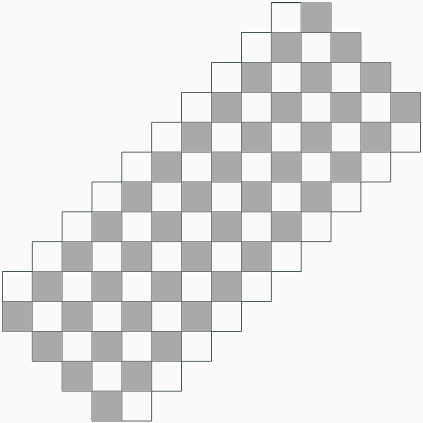

Figure 2: Checkerboard representation of an Aztec Rectangle with $a = 4, b = 10$

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Let $a < b$ be positive integers and $1 \leq s_1 < s_2 < \cdots < s_a \leq b$. Then the number of domino tilings of $\mathcal{AR}_{a,b}$ where all unit squares from the south-eastern side are removed except for those in positions $s_1, s_2, \ldots, s_a$ is

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Our goal here is to extend this result.

Graphical Condensation

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$$=\mathrm{M}(G-\{w,x\})\,\mathrm{M}(G-\{y,z\})+\mathrm{M}(G-\{w,z\})\,\mathrm{M}(G-\{x,y\}).$$

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- $w, x, y, z \in V_1, |V_1| = |V_2| + 2$; first term vanishes

Sketch Proof

- Superimpose a perfect matching of G (blue) and a perfect matching of $G - \{w, x, y, z\}$ (red) on the same copy of G

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- There is a blue-red alternating path from w to one of x, y, z
- Two such paths cannot cross, so w does not connect to y
- Switch the edges in the path of w and get a pair of matchings of $G - \{w, x\}$ and $G - \{y, z\}$ or of $G - \{w, z\}$ and $G - \{x, y\}$

Pfaffians

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$$\mathrm{Pf}(A) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}.$$

Ciucu's Extension of Kuo's Condensation

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$$\mathrm{M}(G \setminus \{a_1, a_2, \ldots, a_{2k}\}) = \frac{\mathrm{Pf}(A)}{[\mathrm{M}(G)]^{k-1}}. \tag{2.2}$$

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- Then we define $G + W$ as follows: $G + W$ is the induced subgraph of H with vertex set $\mathsf{V}(G + W) = \mathsf{V}(G) \Delta \mathsf{V}(W)$, where Δ denotes the symmetric difference of sets.

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Proof.

Take $n = 2$, $a_1 = w, a_2 = x, a_3 = y, a_4 = z$ and $G = H \setminus \{a_1\}$. □

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The proof of this proposition is similar to an analogous result of Ciucu.

Aztec Rectangles with Defects

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But first we look at the case when $k = 0$.

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- If a_i, a_j are on adjacent sides, then the defects are of different type and we get matchings. This is given in the next result.

Aztec Diamond with defects on adjacent sides

Proposition

Let a, i, j be positive integers such that $1 \leq i, j \leq a$, then the number of domino tilings of $\mathrm{AD}(a)$ with one defect on the southeastern side at the i-th position counted from the south corner and one defect on the northeastern side on the j-th position counted from the north corner as shown in the next figure is given by

$$2^{a(a-1)/2} \binom{a-1}{i-1} \binom{a-1}{j-1} \, {}_3F_2 \left[\begin{matrix} 1, 1-i, 1-j \\ 1-a, 1-a \end{matrix} ; 2 \right].$$

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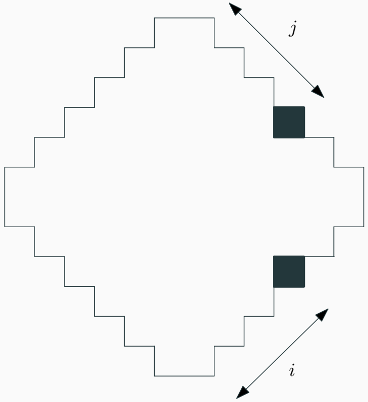

Figure 3: Aztec Diamond with defects on adjacent sides; here $a = 6$, $i = 4$, $j = 4$

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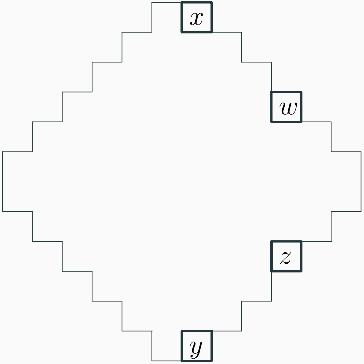

Figure 4: Aztec Diamond with some marked squares; here $a = 6$

Proof contd.

We use induction with respect to a. The base case of induction is $a = 2$. We would also need to check for $i = 1, j = 1, i = a$ and $j = a$ separately.

Proof contd.

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If $a = 2$, then the only possibilities are $i = 1$ or $i = a$ and $j = 1$ or $j = a$, so we do not have to consider this case, once we consider the other mentioned cases.

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If $a = 2$, then the only possibilities are $i = 1$ or $i = a$ and $j = 1$ or $j = a$, so we do not have to consider this case, once we consider the other mentioned cases.

We now note that when either i or j is 1 or a, some dominoes are forced in any tiling and hence we are reduced to an Aztec rectangle of size $(a - 1) \times a$. It is easy to see that our formula is correct for this.

Proof contd.

In the rest of the proof we assume $a \geq 3$ and $1 < i, j < a$. Let us now denote the region we are interested in this proposition as $\mathrm{AD}_a(i,j)$. Using the dual graph of this region and applying Kuo condensation with the vertices as labelled in the previous figure we obtain the following identity (details in the next figure),

Proof contd.

In the rest of the proof we assume $a \geq 3$ and $1 < i, j < a$. Let us now denote the region we are interested in this proposition as $\mathrm{AD}_a(i,j)$. Using the dual graph of this region and applying Kuo condensation with the vertices as labelled in the previous figure we obtain the following identity (details in the next figure),

$$\begin{aligned} \mathrm{M}(\mathrm{AD}_a(i,j))\,\mathrm{M}(\mathrm{AD}(a-1)) = {} & \mathrm{M}(\mathrm{AD}(a))\,\mathrm{M}(\mathrm{AD}_{a-1}(i-1,j-1)) \\ & + \mathrm{M}(\mathcal{AR}_{a-1,a}(j))\,\mathrm{M}(\mathcal{AR}_{a-1,a}(i)). \end{aligned} \tag{3.1}$$

Forced dominoes for different choices of labels

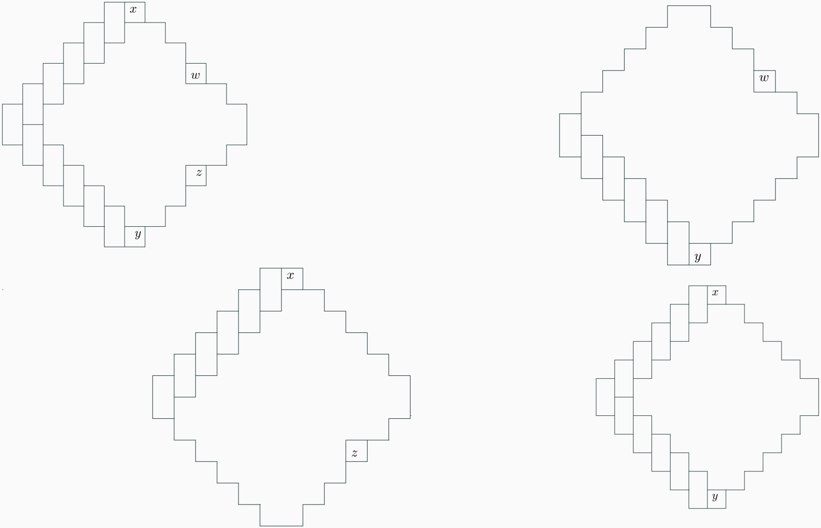

Figure 5: Forced dominoes, where the vertices we remove are marked

Proof contd.

Simplifying the previous equation, we get the following

$$\mathrm{M}(\mathrm{AD}_a(i,j)) = 2^a\,\mathrm{M}(\mathrm{AD}_{a-1}(i-1,j-1)) + 2^{a(a-1)/2}\binom{a-1}{j-1}\binom{a-1}{i-1} \tag{3.2}$$

Proof contd.

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Now, using our inductive hypothesis on this equation we see that we get the expression in the proposition.

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The above follows from using the theorems of Elkies et. al. and Mills et. al.

Now, using our inductive hypothesis on this equation we see that we get the expression in the proposition.

We now look at the general case.

$k > 0$: Aztec Rectangles

- In order to create a region that can be tiled by dominoes we have to remove k more white squares than black squares along the boundary of $\mathcal{AR}_{a,b}$.

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- In order to create a region that can be tiled by dominoes we have to remove k more white squares than black squares along the boundary of $\mathcal{AR}_{a,b}$.
- There are $2b$ white squares and $2a$ black squares on the boundary of $\mathcal{AR}_{a,b}$. We choose $n + k$ of the white squares that share an edge with the boundary and denote them by $\beta_1, \beta_2, \ldots, \beta_{n+k}$ (we will refer to them as defects of type β).

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- We choose any n squares from the black squares which share an edge with the boundary and denote them by $\alpha_1, \alpha_2, \ldots, \alpha_n$ (we refer to them as defects of type α).
- We consider regions of the type $\mathcal{AR}_{a,b} \setminus \{\beta_1, \ldots, \beta_{n+k}, \alpha_1, \ldots, \alpha_n\}$, which are more general than the type considered by other authors.

Preliminaries

We define the region $\mathcal{AR}^k_{a,b}$ to be the region obtained from $\mathcal{AR}_{a.b}$ by adding a string of k unit squares along the boundary of the southeastern side (γ defects) as shown in the figure below.

We define the region $\mathcal{AR}_{a,b}^{k}$ to be the region obtained from $\mathcal{AR}_{a.b}$ by adding a string of k unit squares along the boundary of the southeastern side (γ defects) as shown in the figure below.

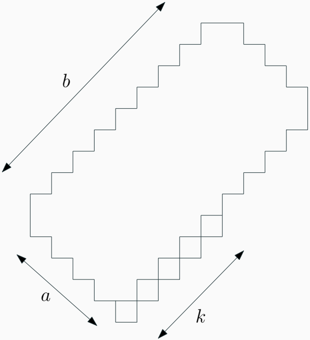

Figure 6: $\mathcal{AR}_{a,b}^{k}$ with $a = 4, b = 8, k = 4$

Main Theorem

Theorem

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Then we have

$$\mathrm{M}(\mathcal{AR}_{a,b} \setminus \{\beta_1, \ldots, \beta_{n+k}, \alpha_1, \ldots, \alpha_n\}) = \frac{1}{[\mathrm{M}(\mathcal{AR}^k_{a,b})]^{n-k+1}} \operatorname{Pf}[(\mathrm{M}(\mathcal{AR}^k_{a,b} \setminus \{\delta_i, \delta_j\}))_{1 \le i < j \le 2n+2k}],$$

where all the terms on the right hand side are given by explicit formulas.

Proof

We need to show that each quantity on the right hand side can be computed.

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- $\mathrm{M}(\mathcal{AR}_{a,b}^{k} \setminus \{\beta_i, \beta_j\}) = 0$,
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- $\mathrm{M}(\mathcal{AR}_{a,b}^{k} \setminus \{\gamma_i, \gamma_j\}) = 0$,
- $\mathrm{M}(\mathcal{AR}_{a,b}^{k} \setminus \{\alpha_i, \gamma_j\}) = 0$,
- $\mathrm{M}(\mathcal{AR}_{a,b}^{k})$ is given by Aztec Diamond Theorem.

Proof contd.

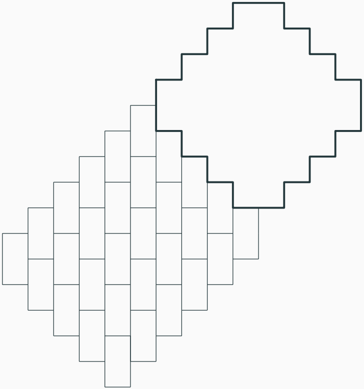


Figure 7: Removing the forced dominoes from $\mathcal{AR}^k_{a,b}$; here $a = 5, b = 10,$ $k = 5$

M($\mathcal{AR}^k_{a,b} \setminus \{\beta_i, \alpha_j\}$)

- It is given by the previous proposition (Aztec Diamond with defects on adjacent sides) if β_i is on the south-eastern side and not above a γ defect;

$M(\mathcal{AR}^k_{a,b} \setminus \{\beta_i, \alpha_j\})$

- It is given by the previous proposition (Aztec Diamond with defects on adjacent sides) if β_i is on the south-eastern side and not above a γ defect;
- Otherwise it is 0,

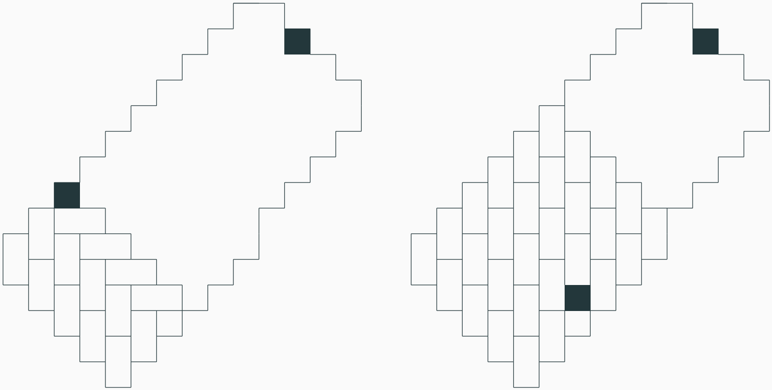
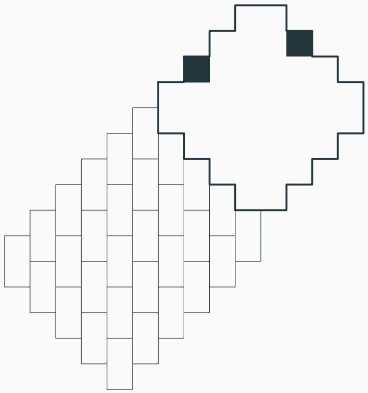


Figure 8: Choices of β-defects that lead to no tiling of $\mathcal{AR}_{a,b}^k$

$\mathrm{M}(\mathcal{AR}_{a,b}^{k} \setminus \{\beta_i, \alpha_j\})$



$\mathrm{M}(\mathcal{AR}^k_{a,b} \setminus \{\beta_i, \gamma_j\})$

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$M(\mathcal{AR}^k_{a,b} \setminus \{\beta_i, \gamma_j\})$

- It is given by Aztec Diamond Theorem if β_i is above a γ defect;
- It is given by the next proposition if the β defect is in the northwestern side at a distance of more than $k-1$ from the western corner;

Proposition

Let $1 \leq a, i \leq b$ be positive integers with $k = b - a > 0$, then the number of domino tilings of $\mathcal{AR}_{a,b}(2, 3, \ldots, k)$ with a defect on the northwestern side in the i-th position counted from the west corner as shown in the Figure 10 is given by

$$2^{a(a+1)/2} \binom{a+k-2}{k-1} \binom{a}{a-i+k} \, {}_3F_2 \left[\begin{matrix} 1, -k-1, i-a-k \\ i-k+1, 2-a-k \end{matrix} ; -1 \right].$$

Regions with defects contd.

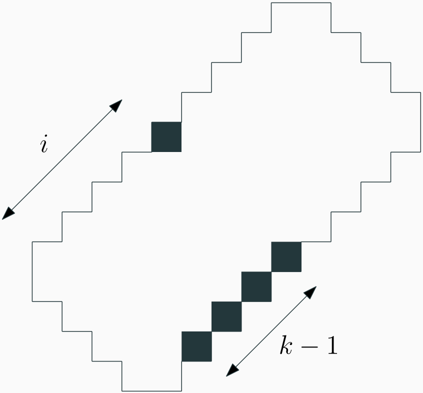

Figure 10: An $a \times b$ Aztec rectangle with defects marked in black; here $a = 4, b = 9.k = 5, i = 5$

M($\mathcal{AR}^k_{a,b} \setminus \{\beta_i, \gamma_j\}$)

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- It is given by the next proposition if the β dent is on the southeastern side;
- Otherwise it is 0.

Proposition

Let $1 \leq a \leq b$ be positive integers with $k = b - a > 0$, then the number of domino tilings of $\mathcal{AR}_{a,b}(j)$ with $k-1$ squares added to the southeastern side starting at the second position (and not at the bottom) as shown in the Figure 11 is given by

$$2^{a(a+1)/2}\binom{a+k-1}{j-1}\binom{j-2}{k-1}\,{}_3F_2\left[\begin{matrix}1, 1-j, 1-k\\ 2-j, 1-a-k\end{matrix};1\right]. \qquad (3.3)$$

Regions with defects contd.

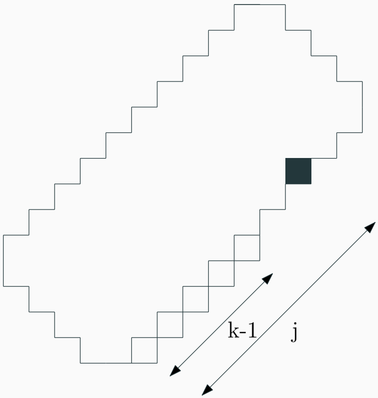

Figure 11: Aztec rectangle with $k-1$ squares added on the southeastern side and a defect on the j-th position shaded in black; here $a = 4, b = 10, k = 6, j = 8$

Remarks

- The proofs of the previous stated propositions, also uses Kuo condensation in various cases.

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- The proofs of the previous stated propositions, also uses Kuo condensation in various cases.
- For the most general case, we have a nested Pfaffian form for the number of matchings.

Theorem

Let $\beta_1, \ldots, \beta_{n+k}$ *be arbitrary defects of type* β *and* $\alpha_1, \ldots, \alpha_n$ *be arbitrary defects of type* α *along the boundary of* $\mathcal{AR}_{a,b}$*. Then* $\mathrm{M}(\mathcal{AR}_{a,b} \setminus \{\beta_1, \ldots, \beta_{n+k}, \alpha_1, \ldots, \alpha_n\})$ *is equal to the Pfaffian of a* $2n \times 2n$ *matrix whose entries are Pfaffians of* $(2k+2) \times (2k+2)$ *matrices of the type in the statement of main theorem.*

Proof

Let $\mathcal{AR}$ be the region obtained from $\mathcal{AR}_{a,b}^k$ by removing k of the squares $\beta_1, \ldots, \beta_{n+k}$. We now apply Ciucu's Theorem to the planar dual graph of $\mathcal{AR}$, with the removed squares chosen to be the vertices corresponding to the n β_i's inside $\mathcal{AR}$ and to $\alpha_1, \ldots, \alpha_n$.

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Now the number of tilings is the Pfaffian of a $2n \times 2n$ matrix with entries of the form $\mathrm{M}(\mathcal{AR} \setminus \{\beta_i, \alpha_j\})$, where β_i is not one of the unit squares that we removed from $\mathcal{AR}_{a,b}^{k}$ to get $\mathcal{AR}$.

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Now the number of tilings is the Pfaffian of a $2n \times 2n$ matrix with entries of the form $\mathrm{M}(\mathcal{AR} \setminus \{\beta_i, \alpha_j\})$, where β_i is not one of the unit squares that we removed from $\mathcal{AR}_{a,b}^k$ to get $\mathcal{AR}$.

We now notice that $\mathrm{M}(\mathcal{AR} \setminus \{\beta_i, \alpha_j\})$ is an Aztec rectangle with all its defects confined to three of the sides. So, we can apply our main theorem and it gives us an expression for $\mathrm{M}(\mathcal{AR} \setminus \{\beta_i, \alpha_j\})$ as the Pfaffian of a $(2k+2) \times (2k+2)$ matrix of the type described in the statement of the main theorem.

Thank you for your attention.