The Remarkable Sequence $1, 2, 7, 42, 429, 7436, \cdots$

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This formula was conjectured by Mills, Robbins and Rumsey to count what are called *alternating sign matrices* (ASMs).

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and the matrix

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A bigger ASM

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$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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the determinant is

$$aei + bfg + cdh - ceg - bdi - afh.$$

A more mathematical definition

More generally, for an $n \times n$ matrix, A with entries $a_{i,j}$ $(1 \leq i, j \leq n)$, the determinant of A is defined as

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

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For the remainder of the talk, A will always denote an $n \times n$ matrix unless otherwise stated.

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Theorem (Desnanot-Jacobi adjoint matrix theorem)

If A is an $n \times n$ matrix, then

$$\det(A)\det(A_{1,n}^{1,n}) = \det(A_1^1)\det(A_n^n) - \det(A_n^1)\det(A_1^n)$$

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$$\det(A) = \frac{1}{\det(A^{1,n}_{1,n})} \times \det\begin{pmatrix} \det(A^1_1) & \det(A^1_n) \\ \det(A^n_1) & \det(A^n_n) \end{pmatrix}.$$

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This gives us a way of evaluating determinants, in terms of smaller determinants.

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$$\det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \frac{1}{a_{2,2}} \times \det \begin{pmatrix} \det \begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} & \det \begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix} \\ \det \begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix} & \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \end{pmatrix}.$$

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$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} = \frac{1}{\det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix}}$$

$$\times \det\begin{pmatrix} \det\begin{pmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} & \det\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \\ \det\begin{pmatrix} a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix} & \det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \end{pmatrix}.$$

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$$\det_{\lambda} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = a_{1,1}a_{2,2} + \lambda a_{2,1}a_{1,2}.$$

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Using the previous observations, they generalized it to an $n \times n$ determinant.

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Theorem (Robbins-Rumsey)

Let A be an $n \times n$ matrix with entries $a_{i,j}$, $\mathcal{A}_n$ be the set of all ASMs, $\mathcal{I}(B)$ be the inversion number of B and $\mathcal{N}(B)$ be the number of -1's in B. Then

$$\det_{\lambda}(A) = \sum_{B \in \mathcal{A}_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{\mathcal{N}(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}}.$$

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This was the first appearance of an ASM in the literature.

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There are seven pairs here whose product is $+1$ and two pairs whose product is -1. So the inversion number is 5.

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Further, if A_n is the number of $n \times n$ ASMs, then $A_{n,1} = A_{n,n} = A_{n-1}$.

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$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)}.$$

This means that the $A_{n.k}$'s are uniquely determined by the $A_{n,k-1}$'s when $k > 1$ and by $A_{n,1} = \sum_{k=1}^{n-1} A_{n-1,k}$.

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What about a proof?

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Shortly after, Greg Kuperberg gave an alternate proof by exploiting a connection between ASMs and statistical physics. It turned out that physicists have been studying ASMs under a different guise.

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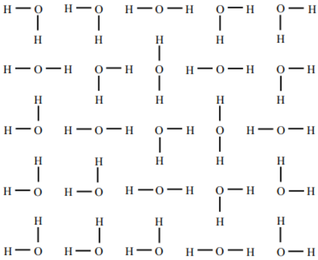

ASMs and Square Ice

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It turned out to be as difficult as enumrating ASMs, and this study was only recently completed in 2016.

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A vertically symmetric ASM

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$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Other objects

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Plane Partitions

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A *plane partition* in an $a \times b \times c$ box is a subset

$$PP \subset \{1, 2, \cdots, a\} \times \{1, 2, \cdots, b\} \times \{1, 2, \cdots, c\}$$

with $(i', j', k') \in PP$ if $(i, j, k) \in PP$ and $(i', j', k') \leq (i, j, k)$.

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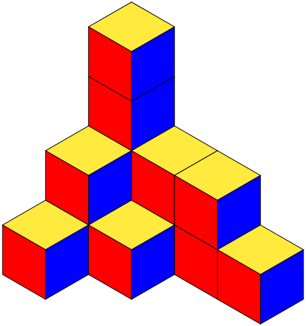

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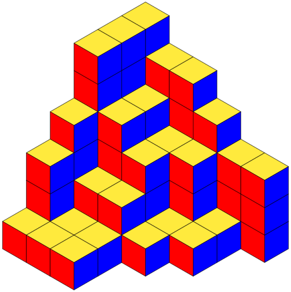

ASMs and Plane Partitions

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Theorem (Ayyer-Behrend-FIscher)

The number of $2n+1 \times 2n+1$ *diagonally and antidiagonally symmetric ASMs with* $n+1$ *occurences of* 1*'s in the diagonals and antidiagonals in their fundamental region is equal to the number of cyclically symmetric plane partitions in an* $n \times n \times n$ *box.*

Diagonally and Antidiagonally Symmetric ASM

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$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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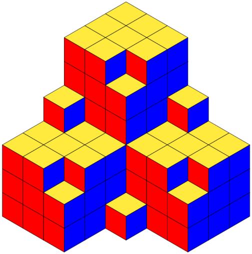

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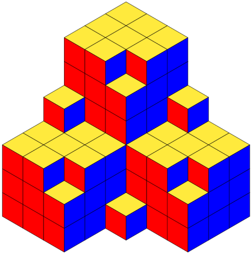

This class of plane partitions inside a $2n \times 2n \times 2n$ box are equinumerous with $n \times n$ ASMs!

Alternating Sign Triangles (ASTs)

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An AST of size n is a triangular array

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$$\begin{array}{ccccc} a_{1,1} & a_{1,2} & \cdots & a_{1,2n-2} & a_{1,2n-1} \\ & a_{2,2} & \cdots & a_{2,2n-2} & \\ & & \vdots & & \\ & & a_{n,n} & & \end{array}$$

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such that

- the entries are either $1, -1$ or 0,
- along the columns and rows the non-zero entries alternate,
- the first non-zero entry from the top is a 1 and the rowsums are equal to 1.

Example

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1 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & 0 & \\
 & & 1 & &
\end{array}
\quad ; \quad
\begin{array}{ccccc}
0 & 0 & 0 & 1 & 0 \\
 & 1 & 0 & 0 & 0 \\
 & & 1 & &
\end{array}
\quad ; \quad
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0 & 0 & 0 & 0 & 1 \\
 & 1 & 0 & 0 & \\
 & & 1 & &
\end{array}
$$

$$
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 & 0 & 0 & 1 & \\
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There are other combinatorial objects which are equinumerous with ASMs, and one of the major open problems in enumerative combinatorics is to find bijections between such objects.

Thank you for your attention!