Refined Enumeration of Alternating Sign Matrices

Manjil P. Saikia (Universität Wien)

Arbeitsgemeinschaft Diskrete Mathematatik, TU Wien

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ASMs

An alternating sign matrix (ASM) of size n is an $n \times n$ matrix with entries in the set $\{0, 1, -1\}$ such that

- all row and column sums are equal to 1,
- and the non-zero entries alternate in each row and column.

For instance, there are 7 ASMs of order 3, these are the six permutation matrices and the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

ASMs Enumeration

Mills, Robbins and Rumsey conjectured that the number of ASMs of size n is given by

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!}$$

or, in product notation

$$\prod_{j=0}^{n-1}\frac{(3j+1)!}{(n+j)!}.$$

This conjecture was later proved by Zeilberger and independently by Kuperberg.

How were they first defined?

Given a matrix A, we let A^i_j denote the matrix that remains when the ith row and jth column of A are deleted. If we remove more than one row or column, then the indices corresponding to those are added to the super- and sub- scripts.

Theorem (Desnanot-Jacobi adjoint matrix theorem)

If A is an $n \times n$ matrix, then

$$\det(A)\det(A^{1,n}_{1,n}) = \det(A^1_1)\det(A^n_n) - \det(A^1_n)\det(A^n_1)$$

or

$$\det(A) = \frac{1}{\det(A^{1,n}_{1,n})} \times \det \begin{pmatrix} \det(A^1_1) & \det(A^1_n) \\ \det(A^n_1) & \det(A^n_n) \end{pmatrix}.$$

This gives us a way of evaluating determinants, in terms of smaller determinants.

More determinants

Reverend Charles L. Dodgson, better known by his pen name of Lewis Carroll used Desnanot-Jacobi theorem to give an algorithm for evaluating determinants in terms of 2×2 determinants.
For instance, we get

$$\det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \frac{1}{a_{2,2}}$$

$$\times \det \begin{pmatrix} \det \begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} & \det \begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix} \\ \det \begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix} & \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \end{pmatrix}.$$

Even more determinants

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} = \frac{1}{\det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix}}$$

$$\times \det\begin{pmatrix} \det\begin{pmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} & \det\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \\ \det\begin{pmatrix} a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix} & \det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \end{pmatrix}.$$

Generalizing the determinant

In the 1980s, Robbins and Rumsey looked at a generalization of the 2×2 determinant, which they called the λ-determinant.
They defined

$$\det_\lambda \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = a_{1,1}a_{2,2} + \lambda a_{2,1}a_{1,2}.$$

Using the previous observations, they generalized it to an $n \times n$ determinant.

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ASMs and Determinants

Their main result in this direction was

Theorem (Robbins-Rumsey)

Let A be an $n \times n$ matrix with entries $a_{i,j}$, $\mathcal{A}_n$ be the set of all ASMs, $\mathcal{I}(B)$ be the inversion number of B and $\mathcal{N}(B)$ be the number of -1's in B. Then

$$\det_\lambda(A) = \sum_{B \in \mathcal{A}_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{\mathcal{N}(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}}.$$

This was the first appearance of an ASM in the literature.

Symmetry Classes of ASMs

In the late 1980's Richard Stanley suggested the study of various symmetry classes of ASMs; this led Robbins to conjecture formulas for many of these classes.

It turned out to be as difficult as enumrating ASMs, and this study was only recently completed in 2016.

Symmetry Classes

- Vertically Symmetric ASMs: $a_{i,j} = a_{i,n+1-j}$, n odd (Kuperberg 2002)
- Half-turn Symmetric ASMs: $a_{i,j} = a_{n+1-i,n+1-j}$, n odd (Razumov-Stroganov 2005), n even (Kuperberg 2002)
- Diagonally Symmetric ASMs: $a_{i,j} = a_{j,i}$, no 'nice' formula
- Quarter-turn Symmetric ASMs: $a_{i,j} = a_{j,n+1-i}$, n odd (Razumov-Stroganov 2005), n even (Kuperberg 2002)
- Horizontally and vertically Symmetric ASMs: $a_{i,j} = a_{i,n+1-j} = a_{n+1-i,j}$, n odd (Okada 2004)
- Diagonally and Antidiagonally Symmetric ASMs: $a_{i,j} = a_{j,i} = a_{n+1-j,n+1-i}$, n odd (Behrend-Fischer-Konvalinka 2017)
- All symmetries: $a_{i,j} = a_{j,i} = a_{i,n+1-j}$, no 'nice' formula.

Symmetry Classes of ASMs contd.

Among these symmetry classes are also the vertically symmetric alternating sign matrices (VSAMSs), which are symmetric under the vertical axis.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Boundary Conditions on ASMs

It is easy to see that there can be only one occurrence of 1 in the first row and column of any ASMs. This also suggests the study of some refined enumeration of these matrices.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Refined Enumeration of ASMs

The study began with conjectures by Robbins about the number of ASMs of order n with the position of the 1 in the first row at the kth column is given by

$$A(n,k) = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{3j+1)!}{(n+j)!}.$$

This was proved by Zeilberger (1996).

Several people have worked on the refined enumration of ASMs as well as their symmetry classes: Behrend, Fischer, Romik, Razumov-Stroganov, Ayyer-Romik, Romik-Karlinsky, etc.

VSASMs

In the case of VASAMs, it turns out that in the second row of such matrices there are exactly two occurrences of 1,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Fischer's Conjecture

Razumov and Stroganov has a formula counting the number of VSASMs with a fixed one in the first column.

Ilse Fischer had conjectured that the number of $(2n+1) \times (2n+1)$ VSASMs, where the first one in the second row is in the ith column is equal to

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!} \left(\prod_{j=1}^{n-1} \frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!} \right). \tag{1}$$

We will prove this conjecture.

Bijection between ASMs and Six Vertex Model

Kuperberg's proof of the ASM conjecture was by exploiting a bijection between the ASMs and a model in statistical physics, called the six-vertex model.

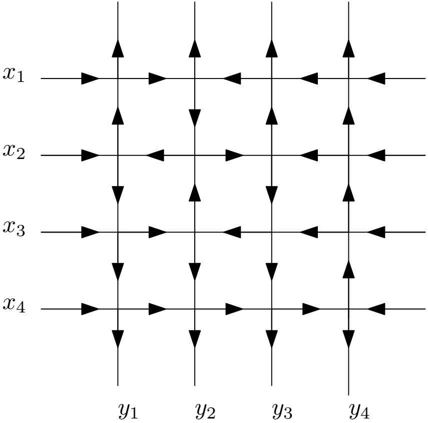


Figure: Six Vertex Model with Domain Wall Boundary Condition.

Bijection between ASMs and Six Vertex Model

A state of a corresponding six-vertex model is an orientation on the edges of this graph, such that both the in-degree and the out-degree of each vertex with degree 4 is 2.

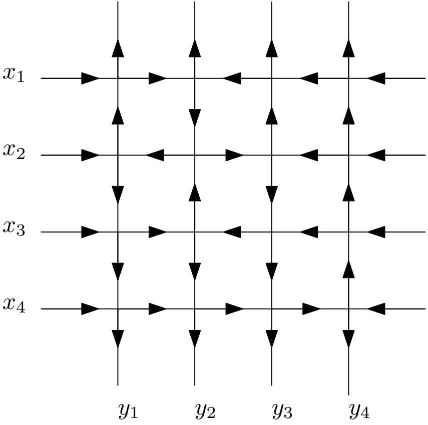


Figure: Six Vertex Model with Domain Wall Boundary Condition.

The Bijection

If we associate to each of the degree 4 vertex in a six-vertex state with a number, as given in the figure below

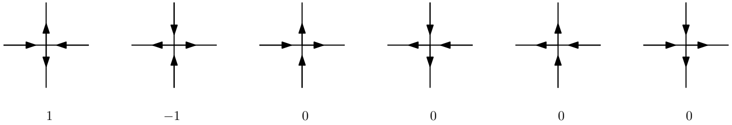


Figure: The corresponding states of the six-vertex model and the entries of an ASM.

then we obtain a matrix with entries in the set $\{0, 1, -1\}$.

Such a matrix will be an ASM, and we get a bijection between ASMs and states of the six-vertex model.

Example

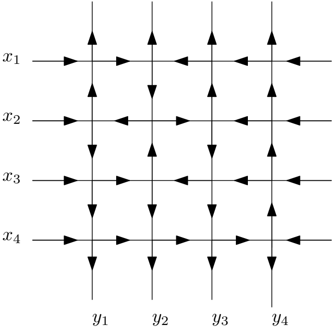


Figure: Six Vertex Model with Domain Wall Boundary Condition.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Weighted Enumeration

- We assign to each vertex v, a weight $w(v)$.
- Weight of a state C is $W(C) = \prod_{v \in C} w(v)$.
- Generating function or the partition function $Z_n = \sum_C W(C)$.
- Specializing the parameters in Z_n, we get enumeration results.

Our Weights

q is a parameter, which we will specialize later, $\overline{x} = \dfrac{1}{x} = x^{-1}$ and $\sigma(x) = x - \overline{x}$.

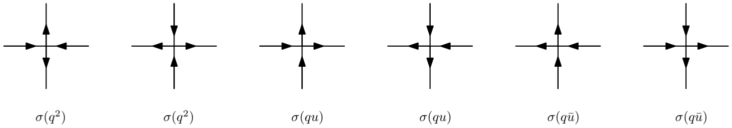
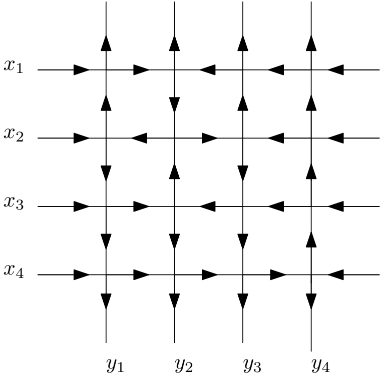


Figure: The weights of the vertices of an ASM with spectral parameter u.

How are weights assigned?



A vertex lying at the intersection of a vertical line with parameter y_j and a horizontal line with parameter x_i is assigned the weight $\dfrac{x_i}{y_j}$.

Another type of ASM

In order to study VSASMs we need what are called U-turn domain boundary wall conditions.

The set of VASAMs is a subset of what are called U-turn ASMs or ASMs with U-turn boundary (UASMs).

We will explain this connection shortly.

UASMs

An U-turn ASM is an $2n \times n$ array which satisfies the usual properties of ASMs if one looks at it vertically.

However, if one looks at it horizontally then the 1's and -1's alternate if we start along an odd numbered row from left to right and then continue along the next even numbered row from right to left.

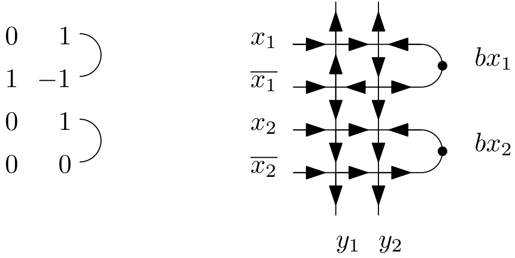


Figure: An U-turn ASM with the corresponding six-vertex state.

New weights

As can be seen from the figure, we add an additional parameter on the U-turns.

This gives rise to two new type of vertices whose corresponding weights are given below.

$\sigma(bu)$ $\sigma(b\overline{u})$

Figure: Weights of the new vertices.

Partition Function

Tsuchiya was the first to consider a U-turn domain wall boundary condition, and gave a partition function for them.

$$Z_U(n; \mathbf{x}, \mathbf{y}) = \frac{\sigma(q^2)^n \prod_i (\sigma(b\overline{y_i})\sigma(q^2 x_i^2)) \prod_{i,j} (\sigma'(x_i\overline{y_j})\sigma'(x_i y_j))}{\prod_{i<j} (\sigma(\overline{x_i} x_j)\sigma(y_i \overline{y}_j)) \prod_{i\leq j} (\sigma(\overline{x_i x_j})\sigma(y_i y_j))} \times \det M_U(n; \mathbf{x}, \mathbf{y}), \tag{2}$$

where $\sigma'(x) = \sigma(qx)\sigma(q\overline{x})$ and M_U is an $n \times n$ matrix defined as

$$M_U(n; \mathbf{x}, \mathbf{y})_{i,j} = \frac{1}{\sigma'(x_i \overline{y}_j)} - \frac{1}{\sigma'(x_i y_j)}.$$

Refined Enumeration of USASMs

We assume

- The first U-turn is downward pointing.
- And the first row has no 1's.

Let $A_U(2n+1, i; w, s)$ be the total weight of the UASM satisfying these conditions

- unique 1 in the second row is in the ith column,
- and each -1 has a multiplicative weight of w,
- and each upward orientation of an U-turn has multiplicative weight s.

Let us denote the partition function of such a configuration to be $Z_U(n; \mathbf{x}, \mathbf{y})$.

We will consider the case for $\mathbf{x} = (x, 1, \ldots, 1)$ and $\mathbf{y} = \mathbf{1}$.

Refined Enumeration of UASMs

If we consider only the first and second row of such a UASM, we see that the total weight of these rows will be

$$\sigma(qx)^n \sigma(q\bar{x})^{i-1} \sigma(q^2) \sigma(qx)^{n-i} \sigma(bx).$$

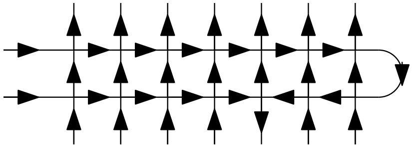


Figure: Weights of the first two.

Refined Enumeration of UASMs

If we consider only the first and second row of such a UASM, we see that the total weight of these rows will be

$$\sigma(qx)^n\sigma(q\bar{x})^{i-1}\sigma(q^2)\sigma(qx)^{n-i}\sigma(bx).$$

Let the number of -1's in the matrix under consideration be k and the number of upward turns be l, then we will see that the total weight of such a configuration would be

$$\sigma(q)^{2n^2-3n+2}\sigma(q^2)^n\sigma(qx)^{2n-1}\sigma(qb)^n \times \left(\frac{\sigma(q^2)}{\sigma(q)}\right)^{2k}\left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{i-1}\left(\frac{\sigma(\bar{q}b)}{\sigma(qb)}\right)^{l}. \tag{3}$$

Refined Enumeration of UASMs

From this we have the following

$$\sum_{i=1}^{n} A_U(2n+1, i; w, s)t^{i-1} = \frac{Z_U(n; (w, 1, \ldots, 1), \mathbf{1})}{\sigma(q)^{2n^2-3n+2}\sigma(q^2)^n\sigma(qx)^{2n-1}\sigma(qb)^n}, \tag{4}$$

where $w = \left(\frac{\sigma(q^2)}{\sigma(q)}\right)^2$, $s = \frac{\sigma(\overline{q}b)}{\sigma(qb)}$ and $t = \frac{\sigma(q\overline{x})}{\sigma(qx)}$.

Properties of Z_U

The partition function had been studied by Razumov and Stroganov. They consider the modified partition function

$$Z(n; \mathbf{x}, \mathbf{y}) = \frac{Z_U(n; \mathbf{x}, \mathbf{y})}{\prod_i (\sigma(b\overline{y_i})\sigma(q^2 x_i^2))}.$$

Let $f_U(n; u) = \sigma(u)^{4n-2}\sigma(u^2)Z(n; u, 1, \ldots, 1))$.

Properties of f

We have the following,

Lemma

1. *The function* $u^{6n-2} f_U(n; u)$ *is a polynomial of degree* $6n-2$ *in* u^2.
2. *The function* $f_U(n; u)$ *satisfies the relation*
$$f_U(n; \overline{u}) = -f_U(n; u).$$
3. *The function* $f_U(n; u)$ *satisfies the relation*
$$f_U(n; u) + f_U(n; a^2 u) + f_U(n; a^4 u) = 0.$$
4. *The Laurent polynomial* $f_U(n; u)$ *is divisible by* $\sigma(u)^{4n-2}$ *and by* $\sigma(u^2)$.

Properties of Z_U

Lemma

The function Z is proportional to the function

$$\varphi(2n,u) = \frac{(-1)^{2n-1}}{\sigma(q)\binom{4n-2}{2n-1}} \sum_{k=0}^{2n-1} \binom{2n-4/3}{2n-1-k}\binom{2n-2/3}{k} \times \sigma(u^{6n-2-6k}). \tag{5}$$

We also have the following

$$\frac{1}{A(n-1)} \sum_{i=1}^{n} A(n,i) t^{i-1} = \frac{\sigma(q)^{3n-2}\varphi(n;u)}{\sigma(qu)^{n-1}\sigma(u)^{2n-1}}, \tag{6}$$

where $A(n)$ is the number of $n \times n$ ASMs, and $A(n,i)$ is the number of $n \times n$ ASMs with the first 1 in the first row at the ith position.

Properties of Z_U

If $q = \exp(i\pi/3)$, then $\sigma(q^2) = \sigma(q)$ and so $w = 1$, we have

$$\frac{1}{A_U(2n-2;1,s)} \sum_{i=1}^{n} A_U(2n,i;1,s) t^{i-1} = \frac{\sigma(q)^{6n-3}\sigma(\overline{q}b)\sigma(b)\varphi(2n;u)}{2\sigma(u)^{4n-2}\sigma(u^2)\sigma(qu)^{2n-1}\sigma(qb)} \tag{7}$$

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Generating Function

Combining everything from above, we get

Theorem

$$\frac{1}{A_U(2n-2;1,s)}\sum_{i=1}^{n} A_U(2n,i;1,s)t^{i-1} = \frac{1}{A(2n-1)}\frac{t+s}{t}\sum_{i=1}^{n} A(2n,i)t^{i-1}, \tag{8}$$

where $s = \dfrac{\sigma(\overline{q}b)}{\sigma(qb)}$ *and* $t = \dfrac{\sigma(q\overline{x})}{\sigma(qx)}$.

Some observations

- VSASMs occur only for odd order.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Some observations

- VSASMs occur only for odd order.
- We need only the first $n+1$ columns of the VSASM to know the full matrix.
- The middle column is an alternating row with 1 and -1.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Some observations

- VSASMs occur only for odd order.
- We need only the first $n+1$ columns of the VSASM to know the full matrix.
- The middle column is an alternating row with 1 and -1.
- So, n columns are sufficient to know the whole matrix.
- Moreover, the first and last rows are always the same.

Transformation

We can transform a VSASM into an USASM in two steps:

- Delete the last row.
- Connect pairwise the alternating edges on the right most column of the $2n \times n$ matrix.

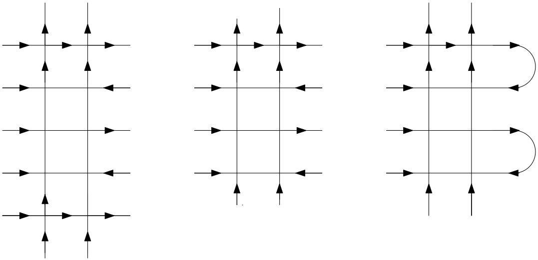

Figure: Transformation of a VSASM into an UASM.

Notice that all U-turns are downward pointing.

Proof

Theorem

The number of $(2n+1)\times(2n+1)$ *VSASM with a* 1 *in the* i*-th position in it's second row is given by*

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!}\left(\prod_{j=1}^{n-1}\frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!}\right). \tag{9}$$

For any VSASM, we shall have $s=0$,and we had the following

$$\frac{1}{A_U(2n-2;1,s)}\sum_{i=1}^{n}A_U(2n,i;1,s)t^{i-1} = \frac{1}{A(2n-1)}\frac{t+s}{t}\sum_{i=1}^{n}A(2n,i)t^{i-1}, \tag{10}$$

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Proof

So, we have $A_U(2n-2;1,0) = A_V(2n-1)$, the number of VSASMs of order $2n-1$.

$A_U(2n,i;1,0) = A_V(2n+1,i)$, the number of VSASMs of order $2n+1$ whose first occurence of 1 in the second row is at the ith column.

So, we get

$$\frac{1}{A_V(2n-1)}\sum_{i=1}^{n} A_V(2n+1,i)t^{i-1} = \frac{1}{A(2n-1)}\sum_{i=1}^{n} A(2n,i)t^{i-1}, \tag{11}$$

Proof

This gives us

$$A_V(2n+1,i) = \frac{A_V(2n-1)A(2n,i)}{A(2n-1)}.$$

The quantities on the right are all known, and we get the result.

Remarks

In the light of the above, as already pointed out by Fischer, we also have the following result.

Theorem

$$A_V(2n+1;i) = A'_V(2n+1;i) + A'_V(2n+1;i+1),$$

where $A'_V(2n+1;i)$ are the number of $2n+1 \times 2n+1$ VSASM with the unique 1 in the first column at the i-th position.

A bijective proof of this result would be very interesting.

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Other Refinements

Other classes of ASMs are the quarter-turn symmetric ASMs (QTASMs) and half-turn symmetric ASMs (HTASMs), Kuperberg and Razumov-Stroganov have enumerated these class of ASMs.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

QTASMs

The following is true for QTASMs,

Theorem

$$A_{QT}(4n+\epsilon; i, w) = wA^2(n; i, w)A_{HT}(2n+\epsilon; i, w),$$

where

- $A_{QT,HT}(m; i)$ *are the number of quarter-turn and half-turn symmetric ASMs of order* m *where the position of the* 1 *in the first row is at the* i*th column*
- *and* w^i *is a multiplicative weight assigned to each such ASM with respect to the position of the* 1 *in the first row;*
- *and* $\epsilon \in \{0, 1, -1\}$.

This gives refined enumeration results for these ASMs.

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qQTSAMs

Notice that $\epsilon = 2$ was missing in the previous class.

This is because QTSASMs do not exit for order $4k + 2$.

But for this order Duchon defined a quasi QTSASM (qQTSASM) which is exactly like a QTSASM except for the middle four entries $(a_{2n,2n}, a_{2n+1,2n}, a_{2n,2n+1}, a_{2n+1,2n+1})$.

These entries can be either $(0, -1, -1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

qQTSAMs

Notice that $\epsilon = 2$ was missing in the previous class.

This is because QTSASMs do not exit for order $4k+2$.

But for this order Duchon defined a quasi QTSASM (qQTSASM) which is exactly like a QTSASM except for the middle four entries $(a_{2n,2n}, a_{2n+1,2n}, a_{2n,2n+1}, a_{2n+1,2n+1})$.

These entries can be either $(0,-1,-1,0)$ or $(1,0,0,1)$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

qQTSASMs

Duchon conjectured the number of qQTSASMs, Aval and Duchon proved this conjecture.

Duchon also conjectured something similar to the weighted refined enumeration of QTSASMs, which is now the following theorem.

Theorem

$$A_{qQT}(4n+2; i, w) = wA(n; i, w)A(n+1; i, w)A_{HT}(2n+1; i, w),$$

where

- $A_{qQT}(m; i)$ *are the number of quasi-quarter-turn symmetric ASMs of order* m *where the position of the* 1 *in the first row is at the* ith *column*
- *and* w^i *is a multiplicative weight assigned to each such ASM with respect to the position of the* 1 *in the first row.*

Thank you for your attention!