Graphical Condensation and Counting Perfect Matchings of Planar Graphs

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$$\mathrm{M}(G)\,\mathrm{M}(G-\{w,x,y,z\})+\mathrm{M}(G-\{w,y\})\,\mathrm{M}(G-\{x,z\})$$

$$=\mathrm{M}(G-\{w,x\})\,\mathrm{M}(G-\{y,z\})+\mathrm{M}(G-\{w,z\})\,\mathrm{M}(G-\{x,y\}).$$

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- $w, x \in V_1, y, z \in V_2, |V_1| = |V_2|$; third term vanishes
- $w, x, y, z \in V_1, |V_1| = |V_2| + 2$; first term vanishes

Sketch Proof

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- There is a blue-red alternating path from w to one of x, y, z
- Two such paths cannot cross, so w does not connect to y
- Switch the edges in the path of w and get a pair of matchings of $G - \{w, x\}$ and $G - \{y, z\}$ or of $G - \{w, z\}$ and $G - \{x, y\}$

Pfaffians

Let $A = (a_{i,j})$ be a $2n \times 2n$ antisymmetric matrix and Γ_n be the set of all perfect matchings of K_{2n}.

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$$\mathrm{Pf}(A) = \sum_{\pi=\{(i_1,j_1),\ldots,(i_n,j_n)\}\in\Gamma_n} \operatorname{sgn}\pi \prod_{k=1}^{n} a_{i_k,j_k}$$

where $\operatorname{sgn}\pi = \operatorname{sgn} i_1 j_1 i_2 j_2 \ldots i_n j_n$.

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$$\mathrm{Pf}(A) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}.$$

Ciucu's Extension of Kuo's Condensation

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Let G be a planar graph with the vertices $a_1, a_2, \ldots, a_{2k}$ appearing in that cyclic order on a face of G. Consider the skew-symmetric matrix $A = (a_{ij})_{1 \leq i,j \leq 2k}$ with entries given by

$$a_{ij} := \mathrm{M}(G \setminus \{a_i, a_j\}), \text{if } i < j. \tag{0.1}$$

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Then we have that

$$\mathrm{M}(G \setminus \{a_1, a_2, \ldots, a_{2k}\}) = \frac{\mathrm{Pf}(A)}{[\mathrm{M}(G)]^{k-1}}. \tag{0.2}$$

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- Let H be a planar graph and G be an induced subgraph of H and let $W \subseteq \mathsf{V}(H)$.
- Then we define $G + W$ as follows: $G + W$ is the induced subgraph of H with vertex set $\mathsf{V}(G + W) = \mathsf{V}(G) \Delta \mathsf{V}(W)$, where Δ denotes the symmetric difference of sets.

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Corollary (Eric Kuo)

Let $G = (V_1, V_2, E)$ be a bipartite planar graph with $|V_1| = |V_2| + 1$; and let w, x, y and z be vertices of G that appear in cyclic order on a face of G.

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Proof.

Take $n = 2$, $a_1 = w, a_2 = x, a_3 = y, a_4 = z$ and $G = H \setminus \{a_1\}$. $\square$

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Let H be a planar graph and G be an induced subgraph of H with the vertices $a_1, \ldots, a_{2k}$ appearing in that cyclic order among the vertices of some face of H.

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Proposition

Let H be a planar graph and G be an induced subgraph of H with the vertices $a_1, \dots, a_{2k}$ appearing in that cyclic order among the vertices of some face of H. Then

$$\mathrm{M}(G)\,\mathrm{M}(G+\{a_1,\dots,a_{2k}\}) + \sum_{l=2}^{k} \mathrm{M}(G+\{a_1,a_{2l-1}\})\,\mathrm{M}(G+\overline{\{a_1,a_{2l-1}\}}) = \sum_{l=1}^{k} \mathrm{M}(G+\{a_1,a_{2l}\})\,\mathrm{M}(G+\overline{\{a_1,a_{2l}\}}), \tag{0.5}$$

where $\overline{\{a_i,a_j\}}$ stands for the complement of $\{a_i,a_j\}$ in the set $\{a_1,\dots,a_{2k}\}$.

Proof of the Proposition

We recast the equation in terms of disjoint unions of cartesian products as follows

$$\mathcal{M}(G) \times \mathcal{M}(G + \{a_1, \ldots, a_{2k}\}) \cup \mathcal{M}(G + \{a_1, a_3\}) \times \mathcal{M}(G + \overline{\{a_1, a_3\}}) \\ \cup \ldots \cup \mathcal{M}(G + \{a_1, a_{2k-1}\}) \times \mathcal{M}(G + \overline{\{a_1, a_{2k-1}\}}) \tag{0.6}$$

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where $\mathcal{M}(F)$ denotes the set of perfect matchings of the graph F.

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- For each element (μ, ν) of (0.6) or (0.7), we think of the edges of μ as being marked by solid lines and that of ν as being marked by dotted lines, on the same copy of the graph H.

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- Thus, the total weight of the elements in the set (0.6) is same as the left hand side of equation (0.5) and the total weight of the elements in the set (0.7) equals the right hand side of equation (0.5).

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- Thus, the total weight of the elements in the set (0.6) is same as the left hand side of equation (0.5) and the total weight of the elements in the set (0.7) equals the right hand side of equation (0.5).
- To prove our result, we have to construct a weight-preserving bijection between the sets (0.6) and (0.7).

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- If $(\mu, \nu) \in \mathcal{M}(G) \times \mathcal{M}(G + \{a_1, \ldots, a_{2k}\})$ we note that when considering the edges of μ and ν together on the same copy of H, each of the vertices $a_1, \ldots, a_{2k}$ is incident to precisely one edge (either solid or dotted depending on the graph G and the vertices a_i's),

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- Thus $\mu \cup \nu$ is the disjoint union of paths connecting the a_i's to one another in pairs, and cycles covering the remaining vertices of H.

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- Thus $\mu \cup \nu$ is the disjoint union of paths connecting the a_i's to one another in pairs, and cycles covering the remaining vertices of H.
- We now consider the path containing a_1 and change a solid edge to a dotted edge and a dotted edge to a solid edge. Let this pair of matchings be (μ', ν').

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- Also, we note that the end edges of this path will be either dotted or solid depending on our graph G and the vertices a_i's. So (μ', ν') is an element of (0.7).

Proof contd.

- If $(\mu, \nu) \in \mathcal{M}(G + \{a_1, a_3\}) \times \mathcal{M}(G + \overline{\{a_1, a_3\}})$, then we map it to a pair of matchings (μ', ν') obtained by reversing the solid and dotted edges along the path in $\mu \cup \nu$ containing a_3.

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- With a similar reasoning like above, this path must connect a_3 to one of the even-indexed vertices and a similar argument will show that indeed (μ', ν') is an element of (0.7).

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- With a similar reasoning like above, this path must connect a_3 to one of the even-indexed vertices and a similar argument will show that indeed (μ', ν') is an element of (0.7).
- If $(\mu, \nu) \in \mathcal{M}(G + \{a_1, a_{2i+1}\}) \times \mathcal{M}(G + \overline{\{a_1, a_{2i+1}\}})$ with $i > 1$, we have the same construction with a_3 replaced by a_{2i+1}.

Proof contd.

- The map $(\mu, \nu) \mapsto (\mu', \nu')$ is invertible because given an element in (μ', ν') of (0.7), the pair (μ, ν) that is mapped to it is obtained by shifting along the path in $\mu' \cup \nu'$ that contains the vertex a_{2i}, such that $(\mu', \nu') \in \mathcal{M}(G + \{a_1, a_{2i}\}) \times \mathcal{M}(G + \overline{\{a_1, a_{2i}\}})$.

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Now we can prove our result.

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For the induction step, we assume that the statement holds for $k - 1$ with $k \geq 2$.

Proof of Theorem

Let A be the matrix

$$\begin{pmatrix} 0 & \mathsf{M}(G+\{a_1,a_2\}) & \mathsf{M}(G+\{a_1,a_3\}) & \cdots & \mathsf{M}(G+\{a_1,a_{2k}\}) \\ -\mathsf{M}(G+\{a_1,a_2\}) & 0 & \mathsf{M}(G+\{a_2,a_3\}) & \cdots & \mathsf{M}(G+\{a_2,a_{2k}\}) \\ -\mathsf{M}(G+\{a_1,a_3\}) & -\mathsf{M}(G+\{a_2,a_3\}) & 0 & \cdots & \mathsf{M}(G+\{a_3,a_{2k}\}) \\ \vdots & \vdots & \vdots & & \vdots \\ -\mathsf{M}(G+\{a_1,a_{2k}\}) & -\mathsf{M}(G+\{a_2,a_{2k}\}) & -\mathsf{M}(G+\{a_3,a_{2k}\}) & \cdots & 0 \end{pmatrix}.$$

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By a well-known property of Pfaffians, we have

$$\mathrm{Pf}(A) = \sum_{i=2}^{2k} (-1)^i \,\mathrm{M}(G+\{a_1,a_i\})\,\mathrm{Pf}(A_{1i}). \tag{0.8}$$

Proof contd.

Now, the induction hypothesis applied to the graph G and the $2k-2$ vertices in $\overline{\{a_i, a_j\}}$ gives us

$$[\mathrm{M}(G)]^{k-2}\,\mathrm{M}(G+\overline{\{a_1, a_i\}}) = \mathrm{Pf}(A_{1i}), \qquad (0.9)$$

where A_{1i} is same as in equation (0.8).

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So using equations (0.8) and (0.9) we get

$$\mathrm{Pf}(A) = [\mathrm{M}(g)]^{k-2}\sum_{i=2} 2k(-1)^i\,\mathrm{M}(G+\{a_1,a_i\})\,\mathrm{M}(G+\overline{\{a_1,a_i\}}). \tag{0.10}$$

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Now using our proposition, we see that the above sum is $\mathrm{M}(G)\,\mathrm{M}(G+\{a_1, \ldots, a_{2k}\})$ and this proves our result.

Where to use the theorem?

The result can be used to enumerate tilings of combinatorial regions like lozenge tilings of a hexagon or domino tilings of Aztec rectangles with holes in them.

Aztec Diamonds

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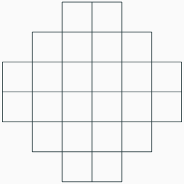

Figure 1: AD(3), Aztec Diamond of order 3

Aztec Diamond Theorem

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Theorem (Elkies–Kuperberg–Larsen–Propp)
The number of domino tilings of an Aztec Diamond of order n is $2^{n(n+1)/2}$.

Aztec Diamond with defects on adjacent sides

Proposition

Let a, i, j be positive integers such that $1 \leq i, j \leq a$, then the number of domino tilings of $\mathrm{AD}(a)$ with one defect on the southeastern side at the i-th position counted from the south corner and one defect on the northeastern side on the j-th position counted from the north corner as shown in the next figure is given by

$$2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1}\ {}_3F_2\left[\begin{matrix}1, 1-i, 1-j\\ 1-a, 1-a\end{matrix}; 2\right].$$

Aztec Diamond with defects on adjacent sides

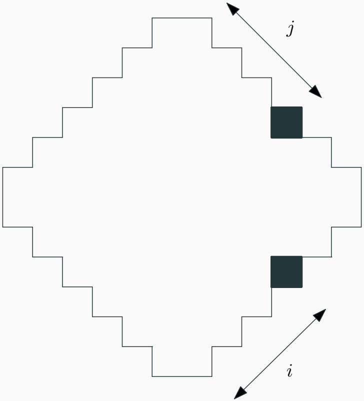

Figure 2: Aztec Diamond with defects on adjacent sides; here $a = 6$, $i = 4$, $j = 4$

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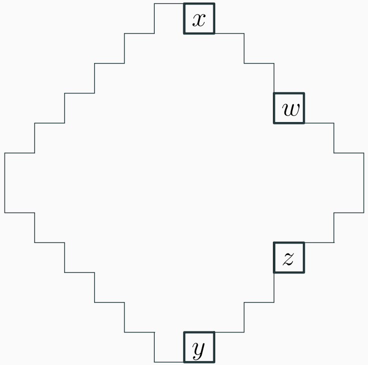

Figure 3: Aztec Diamond with some marked squares; here $a = 6$

Proof contd.

Using the dual graph of this region and applying Kuo condensation with the vertices as labelled in the previous figure we obtain the following identity (details in the next figure),

Proof contd.

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$$\begin{aligned} \mathsf{M}(\mathsf{AD}_a(i,j))\,\mathsf{M}(\mathsf{AD}(a-1)) = {} & \mathsf{M}(\mathsf{AD}(a))\,\mathsf{M}(\mathsf{AD}_{a-1}(i-1,j-1)) \\ & + \mathsf{M}(\mathcal{AR}_{a-1,a}(j))\,\mathsf{M}(\mathcal{AR}_{a-1,a}(i)). \end{aligned} \tag{0.11}$$

Forced dominoes for different choices of labels

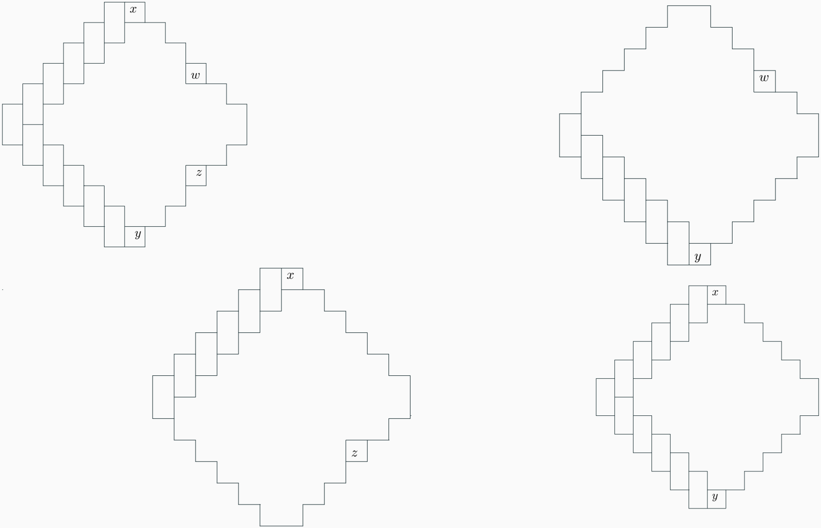

Figure 4: Forced dominoes, where the vertices we remove are marked

Proof contd.

Simplifying the previous equation, we get the following

$$\mathrm{M}(\mathrm{AD}_a(i,j)) = 2^a\,\mathrm{M}(\mathrm{AD}_{a-1}(i-1,j-1)) + 2^{a(a-1)/2}\binom{a-1}{j-1}\binom{a-1}{i-1} \tag{0.12}$$

Proof contd.

Simplifying the previous equation, we get the following

$$\mathrm{M}(\mathrm{AD}_a(i,j)) = 2^a\,\mathrm{M}(\mathrm{AD}_{a-1}(i-1,j-1)) + 2^{a(a-1)/2}\binom{a-1}{j-1}\binom{a-1}{i-1} \tag{0.12}$$

Now, using induction on this equation we see that we get the expression in the proposition.

Thank you for your attention.