Cranks in Ramanujan's Lost Notebook

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Outline

1. Introduction
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   - Crank of a Partition
   - Cranks in Ramanujan's Lost Notebook
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Notations and Preliminaries

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$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \ (a)_\infty := (a;q)_\infty := \lim_{n\infty}(a;q)_n, |\ q\ |< 1.$$

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This satisfies the well-known Jacobi's triple product identity, [3], [9]

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

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We follow [4] and [9] for introductions and properties of $q - Series$.

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Theorem (Ramanujan-Kač-Wakimoto-Evans)

Let $a_k = (-1)^k q^{k(K=1)/2}$. *Then*

$$\frac{(q;q)_\infty^2}{(q/x;q)_\infty (qx;q)_\infty} = \sum_{k=\infty}^{\infty} \frac{a_k(1-x)}{1-xq^k}.$$

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Theorem (Ramanujan)

If $U_n = \alpha^{n(n+1)/2}\beta^{n(n-1)/2}$ *and* $V_n = \alpha^{n(n-1)/2}\beta^{n(n+1)/2}$ *for each integer* n, *then*

$$f(U_1,V_1) = \sum_{k=0}^{N-1} U_k f(\frac{U_{N+k}}{U_k}, \frac{V_{N-k}}{U_k}).$$

Quintuple Product Identity

Theorem (Quintuple Product Identity)

Let $f(a,b)$ be defined as above and let

$$f(-q) := f(-q,-q^2) = (q;q)_\infty.$$

Then

$$f(P^3Q, Q^5/P^3) - P^2 f(Q/P^3, P^3Q^5) = f(-Q^2)\frac{f(-P^2,-Q^2/P^2)}{f(PQ,Q/P)}.$$

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A proof of the above can be found in [8].

Winquist's Identity

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Following the notations given earlier, we have

$$a, q/a, b, q/b, ab, q/(ab), a/b, bq/a, q, q; q)_\infty$$
$$= f(-a^3, -q^3/a^3)\{f(-b^3q, -q^2/b^3) - bf(-b^2q^2, -q/b^3)\}$$
$$-ab^{-1}f(-b^3, -q^3/b^3)\{f(-a^3q, -q^2/a^3) - af(-a^3q^2, -q/a^3)\}.$$

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If n is a positive integer, let $p(n)$ denote the number of unrestricted representations of n as a sum of positive integers, where representations with different orders of the same summands are not regarded as distinct. We call $p(n)$ the partition function.

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Proofs of the above can be found in [3].

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Let $N(m, t, n)$ denote the number of partitions of n with rank congruent to m modulo t. Then Dyson conjectured that

$$N(k, 5, 5n+4) = \frac{p(5n+4)}{5}, \quad 0 \leq k \leq 4,$$

and

$$N(k, 7, 7n+5) = \frac{p(7n+5)}{7}, \quad 0 \leq k \leq 6,$$

which yield combinatorial interpretations of (1) and (2).

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A brief account of these can be found in [11].

Outline

Cranks

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Definition (Crank)

For a partition π, let $\lambda(\pi)$ denote the largest part of π, let $\mu(\pi)$ denote the number of ones in π, and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is then defined to be

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$$c(\pi) = \begin{cases} \lambda(\pi) & \text{if } \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi) & \text{if } \mu(\pi) > 0. \end{cases}$$

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The crank not only leads to combinatorial interpretations of (1) and (2), but also of (3).

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An excellent introduction to cranks is given by Garvan in [12].

Ramanujan and cranks

In page 179 of [13], we find the generating function for cranks (4) in the form

$$F(q) := F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} =: \sum_{n=0}^{\infty} \lambda_n q^n.$$

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$$\lambda_n = \sum_{m=-\infty}^{\infty} M(m,n) a^m.$$

We note that if $a = 1$ in the above, then it reduces to the generating function of $p(n)$.

Dissections

Definition (Dissections)

If

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is any power series, then the m-dissection of $P(q)$ is given by

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We state the results, without proofs.

2-dissection

Theorem (2-dissection)

$$F_a(\sqrt{q}) \equiv \frac{f(-q^3;-q^5)}{(-q^2;q^2)_\infty} + (a-1+\frac{1}{a})\sqrt{q}\frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \ (mod\ a^2+\frac{1}{a^2}). \tag{5}$$

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We note that $\lambda_2 = a^2 + a^{-2}$, which trivially implies that $a^4 \equiv -1 \ (mod\ \lambda_2)$ and $a^8 \equiv 1 \ (mod\ \lambda_2)$. Thus, in (5) a behaves like a primitive 8th root of unity modulo λ_2.

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Thus, if we let $a = exp(2\pi i/8)$ and replace q by q^2 in the definition of a dissection, (5) will give the 2-dissection of $F_a(q)$.

3-dissection

Theorem (3-dissection)

$$F_q(q^{1/3}) \equiv \frac{f(-q^2,-q^7)f(-q^4,-q^5)}{(q^9;q^9)_\infty}$$

$$+(a\text{-}1+1/a)q^{1/3}\frac{f(-q,-q^8)f(-q^4,-q^5)}{(q^9;q^9)_\infty}$$

$$+(a^2+\frac{1}{a^2})q^{2/3}\frac{f(-q,-q^8)f(-q^2,-q^7)}{(q^9;q^9)_\infty} \pmod{a^3+1+\frac{1}{a^3}}.(6)$$

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Again we note that, $\lambda_3 = a^3 + 1 + \frac{1}{a^3}$, from which it follows that $a^9 \equiv -a^6 - a^3 \equiv 1\ (mod\ \lambda_3)$. So in (6), a behaves like a primitive 9th root of unity.

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While if we let $a = exp(2\pi i/9)$ and replace q by q^3 in the definition of a dissection, (6) will give the 3-dissection of $F_a(q)$.

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$$F_a(q)=\frac{f(-q^2,-q^3)}{f^2(-q,-q^4)}f^2(-q^5)-$$

$$4\cos^2(2n\pi/5)q^{1/5}\frac{f^2(-q^5)}{f(-q,-q^4)}+2\cos(4n\pi/5)q^{2/5}\frac{f^2(-q^5)}{f(-q^2,-q^3)}$$
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$$-2\cos(2n\pi/5)q^{3/5}\frac{f(-q,-q^4)}{f^2(-q^2,-q^3)}f^2(-q^5).(7)$$

We observe that (7) has no term with $q^{4/5}$, which is a reflection of (1).

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In contrast to (5) and (6), Ramanujan offererd the 5-dissection in terms of an equality.

Theorem (5-dissection)

$$F_a(q) = \frac{f(-q^2,-q^3)}{f^2(-q,-q^4)} f^2(-q^5) -$$

$$4\cos^2(2n\pi/5)q^{1/5}\frac{f^2(-q^5)}{f(-q,-q^4)} + 2\cos(4n\pi/5)q^{2/5}\frac{f^2(-q^5)}{f(-q^2,-q^3)}$$

$$-2\cos(2n\pi/5)q^{3/5}\frac{f(-q,-q^4)}{f^2(-q^2,-q^3)} f^2(-q^5). (7)$$

We observe that (7) has no term with $q^{4/5}$, which is a reflection of (1).

In fact, one can replace (7) by a congruence and in turn (5) and (6) by equalities. This is done in [5].

Other dissections

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Uniform proofs of these dissections and the others already stated earlier are given in [5].

Other results

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Ramanujan also recorded the first 21 coefficients in the power series of $F_a(q)$ in page 58.
There are many other claims in [13] that are related to cranks. In the remaining period of this project, we shall make an attempt to study them in detail.

The final problem

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It is therefore, very likely that the final problem on which Ramanujan worked was cranks, although it is very unlikely that Ramanujan thought about the combinatorial aspects of cranks.

Outline

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- Generalizations of ranks and cranks.

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Thank You!