Representations of the Symmetric Group

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Outline

1. Introduction to Representation Theory
   - Generalities on Linear Representations
   - Character Theory
2. Representations of S_n
   - Permutation Modules
   - Specht Modules
   - Induced and Restricted Representations
   - Decomposition of M^μ

Some Definitions

Definition

Let G be a finite group with identity 1 and composition defined as $(s,t) \mapsto st$ for some elements s and t in G. A **linear representation** of G in V is a homomorphism

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Definition

An n-dimensional **matrix representation** of a group G is a homomorphism

$$R : G \longrightarrow GL_n(\mathbb{C}),$$

where $GL_n(\mathbb{C})$ is the group of all $n \times n$ matrices over $\mathbb{C}$.

Some Examples

Example

A representation of degree 1 of a group G is a homomorphism $\rho : G \longrightarrow \mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers. Here, since G has finite order the values of $\rho(s)$ are roots of unity. If $\rho(s) = 1$ for all $s \in G$, then this representation is called the **trivial representation**.

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Let the group G act on the finite set X. If V is a vector space whose basis (e_x) is indexed by the elements of X, then the **permutation representation** of G associated with X is given by the linear map $\rho_s : V \longrightarrow V$ where $e_x \mapsto e_{sx}$ and $s \in G$.

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Subrepresentations

Definition

A subspace W of a vector space V is **stable** under the action of a finite group G if $\rho_s w \in W$ for all $s \in G$ and $w \in W$, where $\rho : G \longrightarrow GL(V)$ is a linear representation.

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Definition

Let $\rho : G \longrightarrow GL(V)$ be a linear representation of G in V. Let W be a subspace of V that is stable under the action of G, then $\rho_s\big|_W$ defines a linear representation of G in W, called the **subrepresentation** of V.

Irreducible Representations

Definition

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Theorem

Every representation is a direct sum of irreducible representations.

Character Theory

Definition

Let $\rho : G \longrightarrow GL(V)$ be a linear representation of a finite group G in the vector space V. The complex valued function χ_ρ defined on G by

$$\chi_\rho(s) = \mathrm{Tr}(\rho_s),$$

where $\mathrm{Tr}(\rho_s)$ is the trace of ρ_s, is called the **character** of the representation ρ.

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The **degree** of the character is defined to be the degree of the representation ρ, and the character of an irreducible representation is called the **irreducible** character.

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(ii) $\chi(s^{-1}) = \chi(s)^*$ *for* $s \in G$, *and*

(iii) $\chi(tst^{-1}) = \chi(s)$ *for* $s, t \in G$.

Inner Product

If ϕ and ψ are two complex-valued functions on G, we define their **inner product** by

$$\langle \phi, \psi \rangle = \frac{1}{g} \sum_{t \in G} \phi(t) \psi(t)^*$$

where g is the order of the group G.

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where g is the order of the group G. If χ is the character of a representation of G then we have $\chi(t)^* = \chi(t^{-1})$. So, we have the following for all functions ϕ on G

$$\langle \phi, \chi \rangle = \frac{1}{g} \sum_{t \in G} \phi(t) \chi(t)^* = \frac{1}{g} \sum_{t \in G} \phi(t) \chi(t^{-1}).$$

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(i) *If χ is a character of an irreducible representation then $\langle \chi, \chi \rangle = 1$.*

(ii) *If χ_1 and χ_2 are the characters of two non-isomorphic irreducible representations then $\langle \chi_1, \chi_2 \rangle = 0$.*

Decompositions

Theorem

Let V be a linear representation of G with character ϕ and let V decompose into a direct sum of irreducible representations as follows

$$V = W_1 \oplus \cdots \oplus W_k.$$

Then, if W is an irreducible representation with character χ, the number of W_i's isomorphic to W is equal to $\langle \phi, \chi \rangle$.

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Then, if W is an irreducible representation with character χ, the number of W_i's isomorphic to W is equal to $\langle \phi, \chi \rangle$.

Corollary

The number of W_i's isomorphic to W does not depend on the decomposition.

Character Theory

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Two representations with the same character are isomorphic.

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Theorem

If ϕ is a character of a representation V of G, then $\langle \phi, \phi \rangle > 0$ and it is equal to 1 if and only if V is irreducible.

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Corollary

Every irreducible representation W_i could be obtained from the regular representation with multiplicity equal to the degree of the representation.

An important result

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(i) *The degrees (n_i's) of the irreducible representations W_i's satisfy the formula* $\sum_{i=1}^{k} n_i^2 = g$.

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(i) *The degrees (n_i's) of the irreducible representations W_i's satisfy the formula* $\sum_{i=1}^{k} n_i^2 = g$.

(ii) *If $s \in G$ is not the identity then we have* $\sum_{i=1}^{k} n_i \chi_i(s) = 0$.

Orthonormal Basis

Theorem

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Theorem

The number of irreducible representations of G up to isomorphisms is equal to the number of conjugacy classes of G.

Induced and Restricted Representations

Definition

Let X be the matrix representation of G, then the restriction of X to H denoted by Res^G_H is given by $\mathrm{Res}^G_H(s) = X(s)$ for all $s \in H$. If X has character χ, then we denote the character of the restriction by $\chi \downarrow^G_H$.

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In a similar way we can induce a representation of G from that of H by looking at the matrix representation. In this case we denote the representation by Ind_H^G and the associated character by $\chi \uparrow_H^G$.

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Let H be a subgroup of a finite group G and let ψ and χ be their respective characters. Then we have

$$\langle \psi \uparrow_H^G, \chi \rangle = \langle \psi, \chi \downarrow_H^G \rangle,$$

where the left inner product is on G and the right is on H.

Some Definitions

Definition

A **partition** of a positive integer n is a sequence of positive numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. We write $\lambda \vdash n$ to denote that λ is a **partition** of n.

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A **Young diagram** is a finite collection of boxes (called nodes) arranged in left-justified rows, with the row sizes weakly decreasing.

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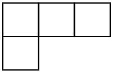
For instance, the Young diagram corresponding to the partition $(3, 1)$ of 4 is given below.



Definitions (contd.)

Definition

A **Young tableau** t **of shape** λ, is a Young diagram of $\lambda \vdash n$ with $1, 2, \ldots, n$ filled in the boxes (nodes) of the Young diagram. In this case, we say that t is a λ-tableau.

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Definition

A **Young tabloid** is an equivalence class of Young tableau under the relation where two tableau are equivalent if each row contains the same elements.

Permutation Module

We observe that S_n acts on the set of λ-tableau: if m is any number in a node of the λ tabloid t then $m\sigma$ is the number in the corresponding node in the new tableau.

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Example

Let $\lambda = (n)$, then we see that M^λ is the vector space generated by the single tabloid with just one row. Since this tabloid is fixed by S_n so, $M^{(n)}$ is the one-dimensional *trivial representation.*

Dimension and Character

Proposition

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, *then*

$$\dim M^\lambda = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_k!}.$$

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Proposition

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *be a partition of* n *and* $g \in S_n$. *Let* $(m_1, m_2, \ldots, m_r)$ *be the cycle type of* g. *Then, the character of the representation of* S_n *on* M^λ *evaluated at* g *is equal to the coefficient of* $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_k^{\lambda_k}$ *in the product*

$$\prod_{i=1}^{r} (x_1^{m_i} + x_2^{m_i} + \cdots + x_k^{m_i}).$$

Character Table

An example of a complete character table obtained from the previous result is given below.

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permutation cycle type	1 $(1,1,1,1)$	(12) $(2,1,1)$	$(12)(34)$ $(2,2)$	(123) $(3,1)$	(1234) (4)
$M^{(4)}$	1	1	1	1	1
$M^{(3,1)}$	4	2	0	1	0
$M^{(2,2)}$	6	2	2	0	0
$M^{(2,1,1)}$	12	2	0	0	0
$M^{(1,1,1,1)}$	24	0	0	0	0

Polytabloids

Definition

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Definition

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$$e_t = \sum_{\pi \in C_t} \text{sgn}(\pi)\pi\{t\}.$$

Lemma

Let t be a tableau and π be a permutation, then $e_{\pi t} = \pi e_t$.

Specht Modules

Definition

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Let $\lambda = (1^n) = (1, 1, \ldots, 1)$ and t be the associated tabloid.
By definition, e_t is the sum of all λ-tabloids multiplied by the sign of the permutation. For any other tabloid t', we have $e_t = \pm e_{t'}$, so we see that here S^λ is the alternating representation.

Specht Modules

Theorem

The Specht modules S^λ forms a complete list of irreducible representations of S_n, where $\lambda \vdash n$.

RSK Correspondence

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- So, our immediate task is to find a suitable basis for S^λ to have some knowledge about its dimension and character.
- We denote by f^λ the number of standard tableau of shape λ. A tableau is called **semi-standard** if the entries are not necessarily distinct, but the rows are weakly increasing and columns are strictly increasing.

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RSK contd.

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Theorem

The RSK correspondence is a bijection between lexicographically ordered $(2 \times n)$ *arrays and pairs of semi-standard tableaux* (s, t) *with shape* λ *for some partition of* n*. If we restrict the arrays to permutations, then the RSK correspondence gives a bijection between elements of* S_n *and pairs of standard tableaux with entries from* $[1, 2, \ldots, n]$.

Consequences of RSK

Corollary

We have

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Corollary

Let $\lambda \vdash n$, then $\dim S^\lambda = f^\lambda$.

Frobenius Character Formula

Theorem (Frobenius Character Formula)

Let $\lambda \vdash n$ with k rows and also let $l_i = \lambda_i + k - i$. Then the character of S^λ on the conjugacy class of a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$, denoted by $\chi_{\lambda\mu}$ is the coefficient of $x_1^{l_1} x_2^{l_2} \ldots x_k^{l_k}$ in the polynomial

$$\prod_{1 \le i < j \le k} (x_i - x_j). \prod_{i=1}^{r} (x_1^{\mu_1} + x_2^{\mu_i} + \cdots + x_k^{\mu_i}).$$

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Theorem

$$\dim S^\lambda = \frac{n!}{\prod_\alpha h_\alpha}.$$

Dominance Order

Definition

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$, then we say that λ dominates μ, written as $\lambda \geq \mu$, if for each $1 \leq i \leq \max(l, r)$ we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i.$$

If $l > r$, we set $\mu_i = 0$ for $i > r$ and vice-versa.

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Lemma (**Dominance Lemma**)

Let t and s be tableaux of shape λ and μ respectively. If for each i, the elements of row i of s are all in different columns of t, then $\lambda \geq \mu$.

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- Let $\lambda \nearrow \mu$ denote that μ can be obtained from λ by adding a box to λ to get a new partition.

Young's lattice

- Much like partitions, Young diagrams can also be partially ordered by inclusion.
- The resulting partially ordered set is called a **Young's lattice**.
- Let $\lambda \nearrow \mu$ denote that μ can be obtained from λ by adding a box to λ to get a new partition.Continuing this way from the empty Young digram we can move upwards adding one box at a time, and get all the partitions of n at the nth level of this lattice.

Branching Rule

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Let $\lambda \vdash n$, then we have

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Again inducing it to S_3 and then finally to S_4 we shall get

$$\mathrm{Ind}_{S_1}^{S_4} S^{(1)} = S^{(4)} \oplus (S^{(3,1)})^3 \oplus (S^{(2,1,1)})^3 \oplus (S^{(2,2)})^2 \oplus S^{(1,1,1,1)}.$$

Kostka Numbers

- We are now interested in finding the number of copies of S^λ in M^μ, and for this we must make use of the Kostka numbers.

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Theorem (Young's Rule)

$$M^\mu \cong \oplus_{\lambda \geq \mu} K_{\lambda\mu} S^\lambda.$$

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Thank You!