Diploma Thesis

Representations of the Symmetric Group

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Dedicated to
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Abstract

In this project, we first study the preliminary results of Representation Theory of Finite Groups (in characteristic zero) and the combinatorics of Young Tableaux. We then use these techniques to explicitly give the irreducible representations of the symmetric group, S_n.

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1

Introduction to Representation Theory

Representation theory is the branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces, as well as modules over these abstract algebraic structures. In essence, a representation makes an abstract algebraic object more concrete by describing its elements with the help of matrices and consequently the algebraic operations in terms of matrix addition and matrix multiplication. The main goal is to represent the group in question in a concrete way.

In this thesis, we shall specifically study the representations of the symmetric group, S_n. To do this, we shall need some preliminary concepts from the general theory of Group Representations which is the motive of this chapter.

1.1 Generalities on Linear Representations

Throughout, we shall assume that V is a vector space over the field $\mathbb{C}$ of complex numbers and denote the set of all invertible linear operators on V by $GL(V)$. This set forms a group with respect to a composition. The material in this chapter has been borrowed from the wonderful expositions of [2], [3] and [4]. [6] has been referred to for some examples. For brevity, in most of the cases we shall either omit the proofs of the results or give only a sketch of the proof. Detailed proofs can be found in the references at the end of this thesis. Unless otherwise mentioned, all groups considered here will be of **finite** order.

Definition 1.1.1. Let G be a finite group with identity 1 and composition defined as $(s,t) \mapsto st$ for some elements s and t in G. A **linear representation** of G in V is a homomorphism

$$\rho : G \longrightarrow GL(V).$$

We shall use the convention of denoting an element $\rho(s)$ by ρ_s where $s \in G$. By definition $\rho_{st} = \rho_s \rho_t$ for all $s, t \in G$, $\rho_1 = 1$ and $\rho_{s^{-1}} = \rho_s^{-1}$. In this thesis, we shall only consider the case when V is finite dimensional. The dimension of V is called the **degree** of the linear representation ρ.

Definition 1.1.2. An n-dimensional **matrix representation** of a group G is a homomorphism

$$R : G \longrightarrow GL_n(\mathbb{C}),$$

where $GL_n(\mathbb{C})$ is the group of all $n \times n$ matrices over $\mathbb{C}$.

Let $\rho : G \longrightarrow GL(V)$ be a linear representation of degree n. Let $(e_1, e_2, \ldots, e_n)$ be a basis for V and R_s be the matrix of ρ_s w.r.t to this basis. Then we have the following:

1. For all $s \in G \rho_s$ is invertible if $\det R_s \neq 0$, and

2. For all $s, t \in G \rho_{st} = \rho_s \rho_t$ if $R_{st} = R_s R_t$.

For $R_s = (r_{ij}(s))$ where $(r_{ij}(s))$ denotes the matrix elements for some i and j the second formula is equivalent to

$$r_{ik}(st) = \sum_j r_{ij}(s) r_{jk}(t).$$

With the above notation, we can go from a linear representation to a matrix representation where R_s is the matrix of ρ_s as defined earlier. Conversely, if we have invertible matrices we can get the other correspondence from this formula.

Definition 1.1.3. Two representations $\rho : G \longrightarrow GL(V)$ and $\rho' : G \longrightarrow GL(V')$ are **isomorphic** if there exists a linear isomorphism $\theta : V \longrightarrow V'$ which satisfies $\theta \circ \rho(s) = \rho'(s) \circ \theta$ for all $s \in G$.

Example 1.1.4. A representation of degree 1 of a group G is a homomorphism $\rho : G \longrightarrow \mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers. Here, since G has finite order the values of $\rho(s)$ are roots of unity. If $\rho(s) = 1$ for all $s \in G$, then this representation is called the **trivial representation**.

Example 1.1.5. Let the group G act on the finite set X. If V is a vector space whose basis (e_x) is indexed by the elements of X, then the **permutation representation** of G associated with X is given by the linear map $\rho_s : V \longrightarrow V$ where $e_x \mapsto e_{sx}$ and $s \in G$. In the final chapter we shall be interested in permutation representations where X is the general symmetric group on n elements. For $X = G$, then this representation is called the **regular representation** of G.

Definition 1.1.6. A subspace W of a vector space V is **stable** under the action of a finite group G if $\rho_s w \in W$ for all $s \in G$ and $w \in W$, where $\rho : G \longrightarrow GL(V)$ is a linear representation.

Definition 1.1.7. Let $\rho : G \longrightarrow GL(V)$ be a linear representation of G in V. Let W be a subspace of V that is stable under the action of G, then $\rho_s\big|_W$ defines a linear representation of G in W, called the **subrepresentation** of V.

For example, if V is a regular representation of G and W is the subspace of dimension 1, then it is a subrepresentation of V which is isomorphic to the trivial representation.

Definition 1.1.8. A linear representation $\rho : G \longrightarrow GL(V)$ is said to be **irreducible** if V is not the zero space and there exists no proper subspace of V that is stable under G.

We notice that the above definition is equivalent to saying that V is not the sum of two representations. A representation of degree 1 is always irreducible. We now come to the first theorem of the theory.

Theorem 1.1.1. *Every representation is a direct sum of irreducible representations.*

The proof of this theorem is by induction on the degree of the representation. Before proceeding to the proof, we work a little with direct sums.

A vector space V is said to be the **direct sum** of its subspaces W and Z if each vector $v \in V$ can be written uniquely as the sum $v = w + z$ for some elements $w \in W$ and $z \in Z$. We then write $V = W \oplus Z$ and say that Z is the complement of W in V. The proof of Theorem 1.1.1 uses the following result whose proof we shall omit.

Theorem 1.1.2. *Let $\rho : G \longrightarrow GL(V)$ be a linear representation of G in V and let W be a vector subspace of V stable under G. Then there exists a complement Z of W in V which is stable under G.*

Now, we can prove Theorem 1.1.1.

Proof. If $\dim(V) = 0$ then there is nothing to prove. So, we assume $\dim(V) = n$, where $n \geq 1$. If V is irreducible then we are done. If V is not irreducible then it has a nontrivial proper subspace W that is stable under G. So by Theorem 1.1.2 we have $V = W \oplus Z$ for Z, the complement of W. Now, we use induction on W and Z to get the result. □

Definition 1.1.9. Let V_1 and V_2 be two vector spaces. The **tensor product** of V_1 and V_2 is a vector space W with a map from $V_1 \times V_2$ into W given by $(x_1, x_2) \mapsto x_1 \otimes x_2$ which satisfies the following conditions:

(i) $x_1 \otimes x_2$ is linear in both the variables, and

(ii) If (e_i) is a basis of V_1 and (e_j) is a basis of V_2 then $(e_i \otimes e_j)$ is a basis of W.

It is clear from this definition that, $\dim(V_1 \otimes V_2) = \dim(V_1).\dim(V_2)$.

Definition 1.1.10. Let $\rho^1 : G \longrightarrow GL(V_1)$ and $\rho^2 : G \longrightarrow GL(V_2)$ be two linear representations of a group G. For $s \in G$, we define $\rho_s \in GL(V_1 \otimes V_2)$ by

$$\rho_s(x_1 \otimes x_2) = \rho_s^1(x_1) \otimes \rho_s^2(x_2)$$

for $x_1 \in V_1$ and $x_2 \in V_2$. Then, we write $\rho_s = \rho_s^1 \otimes \rho_s^2$ and this defines a linear representation of G in $V_1 \otimes V_2$ which is called the tensor product of the given representations.

If in the above $V_1 = V_2 = V$ and θ be an automorphism of $V \otimes V$ such that $\theta(e_i \otimes e_j) = e_j \otimes e_i$ for all pairs (i, j), then we have $\theta(x \otimes y) = y \otimes x$ for $x, y \in V$. Hence, we see that θ is independent of the basis that we choose.

We now define subsets of $V \otimes V$ as follows

$$\mathrm{Sym}^2(V) = \{z \in V \otimes V \mid \theta(z) = z\}$$

and

$$\mathrm{Alt}^2(V) = \{z \in V \otimes V \mid \theta(z) = -z\}$$

which are called the **symmetric power** and the **alternating power**. We can similarly define higher order powers, but for this thesis we consider only second order powers . As an automorphism,θ is linear. Hence, the above two sets are subspaces of $V \otimes V$.

Proposition 1.1.3. *Let $\rho^1 = \rho \otimes \rho$ be a representation of G in $V \otimes V$, then the subspaces $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ are stable under G.*

The proof of this proposition is a simple algebraic argument and is omitted here.

Proposition 1.1.4. *The space $V \otimes V$ decomposes as*

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V).$$

The proof is very elementary, which is omitted here.

1.2 Character Theory

Now we come to a significant part of the theory of representations of a finite group. The characters of a representation that we shall study in this section encodes all the important information about a representation. The results mentioned in this section will be of paramount importance in the next chapter where we find representations of the first few symmetric groups using the content of this chapter.

Definition 1.2.1. Let $\rho : G \longrightarrow GL(V)$ be a linear representation of a finite group G in the vector space V. The complex valued function χ_ρ defined on G by

$$\chi_\rho(s) = \mathrm{Tr}(\rho_s),$$

where $\mathrm{Tr}(\rho_s)$ is the trace of ρ_s, is called the **character** of the representation ρ.

The **degree** of the character is defined to be the degree of the representation ρ, and the character of an irreducible representation is called the **irreducible** character.

Proposition 1.2.1. *If χ is the character of a representation ρ of degree n, then we have the following:*

(i) $\chi(1) = n$,

(ii) $\chi(s^{-1}) = \chi(s)^*$ *for* $s \in G$, *and*

(iii) $\chi(tst^{-1}) = \chi(s)$ *for* $s, t \in G$.

The proof of this result is very simple and we shall omit it here.

Proposition 1.2.2. *Let $\rho^1 : G \longrightarrow GL(V_1)$ and $\rho^2 : G \longrightarrow GL(V_2)$ be two linear representations of G and let χ_1 and χ_2 be their characters. Then we have the following:*

(i) The character χ of the direct sum representation $V_1 \oplus V_2$ is equal to $\chi_1 + \chi_2$, and

(ii) The character χ of the tensor product representation $V_1 \otimes V_2$ is equal to $\chi_1 . \chi_2$.

Proof. (i) Let (e_{i_1}) be a basis of V_1 and R_s^1 be the matrix of ρ_s^1 w.r.t this basis. Similarly, let (e_{i_2}) be a basis of V_2 and R_s^2 be the matrix of ρ_s^2 w.r.t this basis. If we list these bases in order (e_{i_1}, e_{i_2}) we shall get the basis for $V_1 \oplus V_2$. Then the matrix R_s of ρ_s is given by

$$R_s = \begin{bmatrix} R_s^1 & 0 \\ 0 & R_s^2 \end{bmatrix}$$

and from here we shall get the desired result.

(ii) Let $R_s^1 = r_{i_1j_1}(s)$ and $R_s^2 = r_{i_2j_2}(s)$. Then we have

$$\chi_1(s) = \sum_{i_1} r_{i_1j_1}(s)$$

and

$$\chi_2(s) = \sum_{i_2} r_{i_2j_2}(s).$$

It is not very difficult to see that the matrix of the tensor product representation will be $r_{i_1j_1}(s).r_{i_2j_2}(s)$ and hence the character will be given by

$$\chi(s) = \sum_{i_1,i_2} r_{i_1j_1}(s)r_{i_2j_2}(s) = \sum_{i_1} r_{i_1j_1}(s) \sum_{i_2} r_{i_2j_2}(s)$$

and from this the result follows.

□

By using similar arguments with the matrices of the representations and their eigenvalues we can get the following result which will be useful later.

Proposition 1.2.3. *Let $\rho : G \longrightarrow GL(V)$ be a linear representation of G and let χ be its character. Let χ_σ be the character of $\mathrm{Sym}^2(V)$ of V and χ_α be that of $\mathrm{Alt}^2(V)$. Then we have*

$$\chi_\sigma(s) = \frac{1}{2}\left(\chi(s)^2 + \chi(s^2)\right),$$

$$\chi_\alpha(s) = \frac{1}{2}\left(\chi(s)^2 - \chi(s^2)\right)$$

and

$$\chi_\sigma + \chi_\alpha = \chi^2.$$

We are now in a position to state some important results related to linear representations. We begin with the following important theorem.

Theorem 1.2.4 (Schur's Lemma). *Let $\rho^1 : G \longrightarrow GL(V_1)$ and $\rho^2 : G \longrightarrow GL(V_2)$ be two irreducible representations of G and let f be a linear mapping from V_1 to V_2 such that $\rho^2_s \circ f = f \circ \rho^1_s$, for all $s \in G$. Then we have the following:*

(i) If ρ^1 and ρ^2 are not isomorphic, then $f = 0$, and

(ii) If $V_1 = V_2$ and $\rho^1 = \rho^2$, then f is a scalar multiple of the identity.

We shall omit the proof of the above result.

If ϕ and ψ are two complex-valued functions on G, we define their **inner product** by

$$\langle \phi, \psi \rangle = \frac{1}{g} \sum_{t \in G} \phi(t)\psi(t)^*$$

where g is the order of the group G. Clearly, this is linear in ϕ and semilinear in ψ and $\langle \phi, \phi \rangle > 0$ for all $\phi \neq 0$. If χ is the character of a representation of G then we have $\chi(t)^* = \chi(t^{-1})$. So, we have the following for all functions ϕ on G

$$\langle \phi, \chi \rangle = \frac{1}{g} \sum_{t \in G} \phi(t)\chi(t)^* = \frac{1}{g} \sum_{t \in G} \phi(t)\chi(t^{-1}).$$

Theorem 1.2.5. *(i) If χ is a character of an irreducible representation then $\langle \chi, \chi \rangle = 1$.*

(ii) If χ_1 and χ_2 are the characters of two non-isomorphic irreducible representations then $\langle \chi_1, \chi_2 \rangle = 0$.

The proof is a routine use of the definitions and some simple algebraic manipulation and is omitted here. This theorem shows that the irreducible characters form an orthonormal system.

Theorem 1.2.6. *Let V be a linear representation of G with character ϕ and let V decompose into a direct sum of irreducible representations as follows*

$$V = W_1 \oplus \cdots \oplus W_k.$$

Then, if W is an irreducible representation with character χ, the number of W_i's isomorphic to W is equal to $\langle \phi, \chi \rangle$.

The proof follows from definition and Theorem 1.2.5.

Corollary 1.2.7. *The number of W_i's isomorphic to W does not depend on the decomposition.*

Corollary 1.2.8. *Two representations with the same character are isomorphic.*

These results mean that the study of representations can be done by studying their characters instead (which is generally easier to do).

Theorem 1.2.9. *If ϕ is a character of a representation V of G, then $\langle \phi, \phi \rangle > 0$ and it is equal to 1 if and only if V is irreducible.*

The proof is a nice combination of the previous results and is not very difficult. We shall omit the proof here. Next we turn to a short study on regular representations that we defined earlier in this chapter.

Proposition 1.2.10. *The character r_G of a regular representation on a group G of order g is given by*

$$r_G(s) = \begin{cases} g & \text{if } s = 1, \\ 0 & \text{if } s \neq 1. \end{cases}$$

Corollary 1.2.11. *Every irreducible representation W_i could be obtained from the regular representation with multiplicity equal to the degree of the representation.*

Corollary 1.2.12.

(i) The degrees (n_i's) of the irreducible representations W_i's satisfy the formula $\sum_{i=1}^{k} n_i^2 = g$.

(ii) If $s \in G$ is not the identity then we have $\sum_{i=1}^{k} n_i \chi_i(s) = 0$.

We will use these results extensively in the next chapter to calculate the character table of some groups.

The character of the permutation representation that we defined earlier will be called the **permutation character** ahead.

Proposition 1.2.13. *The permutation character is given by the number of elements of the set X that is fixed by s ($s \in G$) that acts on X.*

Proposition 1.2.14. *Let χ be the permutation character of G, then the function $\chi' : G \longrightarrow \mathbb{C}$ defined by $\chi'(s) = \chi(s) - 1$ is also a character of G.*

Since every representation decomposes into irreducibles, so the study of irreducible representation is very important. We focus on this for a while.

Definition 1.2.2. A complex-valued function $f : G \longrightarrow \mathbb{C}$ which is constant on each conjugacy class is called a **class function**.

Proposition 1.2.15. *Let f be a class function on G and let $\rho : G \longrightarrow GL(V)$ be a linear representation of G. Let ρ_f be the linear mapping of V into itself defined by*

$$\rho_f = \sum_{t \in G} f(t)\rho_t.$$

If V is irreducible of degree n and character χ, then ρ_f is a homothety of ratio λ, given by

$$\lambda = \frac{1}{n}\sum_{t \in G} f(t)\chi(t) = \frac{g}{n}\langle f, \chi^* \rangle.$$

Theorem 1.2.16. *The irreducible characters $\chi_1, \ldots, \chi_k$ form an orthonormal basis of the space of class functions.*

Theorem 1.2.17. *The number of irreducible representations of G up to isomorphisms is equal to the number of conjugacy classes of G.*

Proposition 1.2.18. *Let $s \in G$ and let $c(s)$ be the number of elements in the conjugacy class of s. Then* *(i)* $\sum_{i=1}^{k} \chi_i(s)^* \chi_i(s) = \dfrac{g}{c(s)}$.

(ii) For $t \in G$ *which is not conjugate to* s*, we have* $\sum_{i=1}^{k} \chi_i(s)^* \chi_i(t) = 0$.

The proof of the above results are routine and we omit them here.

Let $\chi_1, \ldots, \chi_k$ be the distinct characters of the irreducible representations $W_1, \ldots, W_k$ of G and $n_1, \ldots, n_k$ be their respective degrees. Let $V = Z_1 \oplus \cdots \oplus Z_h$ be a decomposition of V into a direct sum of irreducible representations. For $i = 1, 2, \ldots, k$ let V_i denote the direct sum of those Z_j's that are isomorphic to W_i. Then, we have

$$V = V_1 \oplus \cdots \oplus V_k.$$

This decomposition of V into a direct sum of irreducible representations is called the **canonical decomposition** of V.

1.3 Restricted and Induced Representations

In this short section, we introduce very briefly some special type of representations.

We start with a question. Given a finite group G with a subgroup H, is there a natural way to get representations of G from H and vice-versa? The following definition brings help.

Definition 1.3.1. Let X be the matrix representation of G, then the restriction of X to H denoted by $X \downarrow_H^G$ or Res_H^G is given by $X \downarrow_H^G (s) = X(s)$ for all $s \in H$. If X has character χ, then we denote the character of the restriction by $\chi \downarrow_H^G$.

In a similar way we can induce a representation of G from that of H by looking at the matrix representation. In this case we denote the representation by $X \uparrow_H^G$ or Ind_H^G and the associated character by $\chi \uparrow_H^G$. We come back to this when we do the representations of S_n.

We close this section with the following result that will be used in the final chapter.

Theorem 1.3.1 (Frobenius Reciprocity)**.** *Let H be a subgroup of a finite group G and let ψ and χ be their respective characters. Then we have*

$$\langle \psi \uparrow_H^G, \chi \rangle = \langle \psi, \chi \downarrow_H^G \rangle,$$

where the left inner product is on G and the right is on H.

Proof. Let G and H have orders g and h respectively. We have the following:

$$\begin{aligned}
\langle \psi \uparrow_H^G, \chi \rangle &= \frac{1}{g} \sum_{t \in G} \psi \uparrow_H^G (t) \chi(t^{-1}) \qquad (1.1) \\
&= \frac{1}{gh} \sum_{x \in G} \sum_{t \in G} \psi(x^{-1}tx) \chi(t^{-1}) \\
&= \frac{1}{gh} \sum_{x \in G} \sum_{y \in G} \psi(y) \chi(xy^{-1}x^{-1}) \\
&= \frac{1}{gh} \sum_{x \in G} \sum_{y \in G} \psi(y) \chi(y^{-1}) \\
&= \frac{1}{h} \sum_{y \in G} \psi(y) \chi(y^{-1}) \\
&= \frac{1}{h} \sum_{y \in H} \psi(y) \chi(y^{-1}) \\
&= \langle \psi, \chi \downarrow_H^G \rangle.
\end{aligned}$$

□

2

Irreducible Representations of S_3, S_4 and S_5

In this chapter we compute the character tables of the first few instances of the symmetric group. We will use the results from the previous chapter without much commentary. We follow [2] very closely.

2.1 S_3

We begin with two simple representations of S_3 (which also works for any S_n) namely the *trivial* and *alternating* representation. The trivial representation U is the one where every element is sent into the identity and the alternating representation U' is the one defined by $gv = \text{sgn}(g)v$, where $\text{sgn}(g)$ is the sign of the permutation g.

As there are three conjugacy classes of S_3, referring to Theorem 1.2.17, we know that there is one more irreducible representation of S_3 . So, we consider the *natural representation* of S_3 in which S_3 acts on $\mathbb{C}^3$ by permuting the coordinates. This representation has dimension 3. But there is an invariant subspace of this representation which is generated by $(1, 1, 1)$. Let us consider $V = \{(a, b, c) \in \mathbb{C}^3 \mid a + b + c = 0\}$ which is a representation of dimension 2.

Now, we compute the characters of U and U' before looking at V. The three conjugacy classes of S_3 are the identity with one element, the transpositions represented by (12) with three elements and the three cycles representetd by (123) with two elements. Clearly the character of the trivial representation is 1 for all three conjugacy classes and for the alternating representation it is $1, -1$ and 1 respectively.

Next, we compute χ_V. We have $\mathbb{C}^3 = U \oplus V$, so $\chi_{\mathbb{C}^3} = \chi_U + \chi_V$. Here

$$\chi_{\mathbb{C}^3} = (\mathrm{Tr}(\mathrm{Id}), \mathrm{Tr}(\rho(12)), \mathrm{Tr}(\rho(123))) = (3, 1, 0).$$

This immediately gives us $\chi_V = (2, 0, -1)$. To verify that U, U' and V are indeed the required irreducible representations, we use the orthogonality relations to get the following:

$$\langle \chi_V, \chi_V \rangle = 1$$

and

$$\langle \chi_U, \chi_{U'} \rangle = \langle \chi_U, \chi_V \rangle = \langle \chi_{U'}, \chi_V \rangle = 0.$$

Thus we have the following character table for S_3.

S_3	1	(12)	(123)
U	1	1	1
U'	1	-1	1
V	2	0	-1

2.2 S_4

We work out the character table for S_4 in a similar way. Here we have five conjugacy classes comprising of the identity, six transpositions, eight 3-cycles, six 4-cycles and three products of two disjoint transpositions which will be represented by (12)(345). From the above section, we can readily make the first three irreducible representations U, U' and V given by the following table:

S_4	1	(12)	(123)	(1234)	(12)(345)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1

From Corollary 1.2.12 we have that the sum of squares of the degrees of representations will be equal to order of S_4, which is 24. Let us denote the degrees of the two missing representations as x and y. Then we have

$$1^2 + 1^2 + 3^2 + x^2 + y^2 = 24,$$

which gives us $x = 3$ and $y = 2$ as the only possibility.

Let $V' = V \otimes U'$, then $\chi_{V'} = \chi_V.\chi_{U'} = (3, -1, 0, 1, -1)$. We can verify that this representation is irreducible. Thus, we have four irreducible representations of S_4 and to get the fifth representation, say W, we need to solve the orthogonality relations that we used in the last section. The complete character table is given below.

S_4	1	(12)	(123)	(1234)	(12)(345)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
W	2	0	-1	0	2

2.3 S_5

We close this chapter by computing the character table of S_5. Much of the procedure is like earlier and for the sake of brevity we shall avoid some routine steps.

There are seven conjugacy classes of S_5 corresponding to the seven partitions of 5: the identity, ten transpositions, twenty 3-cycles, thirty 4-cycles, twenty four 5-cycles, fifteen products of two transpositions denoted by (12)(34) and twenty products of a transposition with a 3-cycle denoted by (12)(345). We can compute the characters of U, U', V and V' like earlier to get the following table.

S_5	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1

There are three more representations left. Like before we shall tensor some of the above representations to see if we get any other. We consider $V \otimes V$, but this is equal to $\mathrm{Alt}^2(V) \oplus \mathrm{Sym}^2(V)$. So, we have two possibilities. We compute $\chi_{\mathrm{Alt}^2(V)}$ using Proposition 1.2.3 to get

$$\chi_{\mathrm{Alt}^2(V)} = (6, 0, 0, 0, 1, -2, 0).$$

We see that $\langle \chi_{\mathrm{Alt}^2(V)}, \chi_{\mathrm{Alt}^2(V)} \rangle = 1$ and hence we get one more irreducible representation.

Now, let us denote the dimension of the two remaining representations by x and y and like before after simplification and using Corollary 1.2.12 we have $x^2 + y^2 = 50$. This gives us two possibilities for (x, y). Either it is $(1, 7)$ or $(5, 5)$. But there cannot be any more degree 1 representations of S_5 since then the image of S_5 will be abelian by this representation. But the only normal subgroups of S_5 are $\{1\}$, A_5 and S_5 ,so, the only abelian quotients of S_5 are $S_5/A_5 = \{\pm 1\}$ and $S_5/S_5 = \{1\}$.

Denote one of these degree 5 representations by W. We have

$$\chi_W = (5, a_1, a_2, a_3, a_4, a_5, a_6).$$

If we tensor W with U' we get $W' = W \otimes U'$ with character

$$\chi_{W'} = (5, -a_1, a_2, -a_3, a_4, a_5, -a_6).$$

However if, $W = W'$ then we must have $a_1 = a_3 = a_6 = 0$ which will not be possible due to orthogonality relations. Hence, $W \neq W'$ and by solving a system of linear equations which we shall get from the orthogonality relations we can deduce the values of a_i's for $i = 1, \ldots, 6$. The complete character table for S_5 is given below.

S_5	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1
$\mathrm{Alt}^2(V)$	6	0	0	0	1	-2	0
W	5	1	-1	-1	0	1	1
W'	5	-1	-1	1	0	1	-1

In the next chapter we shall give results for the irreducible representations of the general symmetric group on n elements and ways to compute the character.

3

Representations of S_n

The purpose of this chapter is to present the representations of the symmetric group, S_n focussing on its combinatorial aspects. While developing the theory we shall use the definitions that presented in the first chapter. In the final section we will also show some beautiful interplay between combinatorics and algebra. The material is this chapter is taken from a variety of sources: [1], [3], [7] and [8]. Throughout this chapter we consider the ground field to be of characteristic 0 unless otherwise mentioned.

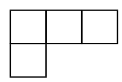
3.1 Preliminaries

Some definitions:

Definition 3.1.1. A **partition** of a positive integer n is a sequence of positive numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. We write $\lambda \vdash n$ to denote that λ is a **partition** of n.

Definition 3.1.2. A **Young diagram** is a finite collection of boxes (called nodes) arranged in left-justified rows, with the row sizes weakly decreasing. The Young diagram associated with the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is the one with l rows and λ_i boxes in the ith row.

For instance, the Young diagram corresponding to the partition $(3, 1)$ of 4 is given below.



Clearly there is a one-to-one correspondence between *partitions* and *Young diagrams*, so we shall use these two terms interchangeably.

Definition 3.1.3. A **Young tableau** t **of shape** λ, is a Young diagram of $\lambda \vdash n$ with $1, 2, \ldots, n$ filled in the boxes (nodes) of the Young diagram, where each number occurs exactly once. In this case, we say that t is a λ-tableau.

For example, all possible tableaux corresponding to the partition $(2, 1)$ are given below:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Definition 3.1.4. A **standard Young tableau** is a Young tableau whose entries are increasing across each row and each column.

The standard tableaux for $(2, 1)$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Definition 3.1.5. A **Young tabloid** is an equivalence class of Young tableau under the relation where two tableau are equivalent if each row contains the same elements.

We observe that S_n acts on the set of λ-tableau: if m is any number in a node of the λ tabloid t then $m\sigma$ is the number in the corresponding node in the new tableau. The new tableau is then $t\sigma$. This action gives an $n!$ dimensional representation of S_n, where elements of the group act on the right.

Definition 3.1.6. Let M^λ be the representation of S_n whose basis is indexed by the set of Young tabloids and the action on the basis is the action on the tabloids.

This is an example of a permutation representation that we discussed earlier. In fact, M^λ is known as the **permutation module** corresponding to λ. Some examples of M^λ

Example 3.1.7. Let $\lambda = (n)$, then we see that M^λ is the vector space generated by the single tabloid with just one row. Since this tabloid is fixed by S_n so, $M^{(n)}$ is the one-dimensional *trivial representation.*

Example 3.1.8. Let $\lambda = (1^n) = (1, 1, \ldots, 1)$, then the λ-tabloid is just a permutation of $\{1, 2, \ldots, n\}$ into n rows and S_n acts on it by acting on the corresponding permutation. Thus, we see here that M^λ is isomorphic to the regular representation which we discussed in the previous chapter.

Example 3.1.9. We now consider $\lambda = (n-1, 1)$, and let $\{t_i\}$ be the λ-tabloid with i in the node of the second row. Then, M^λ has basis $\{t_i\}$'s. The action of $\pi \in S_n$ sends t_i to $t_{\pi(i)}$. So here M^λ is isomorphic to the defining representation which decomposes as the direct sum of the trivial and standard representations.

Proposition 3.1.1. *If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, then*

$$\dim M^\lambda = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_k!}.$$

The proof is very easy once we notice that any permutation that fixes rows preserves the equivalence classes of Young tabloids. So, there are $\lambda_i!$ permutations that permute the ith row and hence we get the result.

Proposition 3.1.2. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n and $g \in S_n$. Let $(m_1, m_2, \ldots, m_r)$ be the cycle type of g. Then, the character of the representation of S_n on M^λ evaluated at g is equal to the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_k^{\lambda_k}$ in the product*

$$\prod_{i=1}^{r}(x_1^{m_i} + x_2^{m_i} + \cdots + x_k^{m_i}).$$

To prove this result, we notice that M^λ is realised as a permutation representation on the tabloids, so its character at an element $\pi \in S_n$ is equal to the number of tabloids that will be fixed by π. The result follows by some generating function arguments. We can also note that Proposition 3.1.1 follows from Proposition 3.1.2 as the dimension is just the value of the character at the identity element.

Example 3.1.10. Now, we compute the characters of the permutation module for S_4 to illustrate the above. From Proposition 3.1.1 we see that the character of $M^{(2,1,1)}$ at identity is 12. We use Proposition 3.1.2 to find the other values, for instance, the character of $M^{(2,2)}$ at the permutation (12) which has cycle type $(2, 1, 1)$ is equal to the coefficient of $x_1^2 x_2^2$ in $(x_1^2 + x_2^2)(x_1 + x_2)^2$ and so on. The complete character table thus obtained is given below.

permutation cycle type	1 $(1,1,1,1)$	(12) $(2,1,1)$	(12)(34) $(2,2)$	(123) $(3,1)$	(1234) (4)
$M^{(4)}$	1	1	1	1	1
$M^{(3,1)}$	4	2	0	1	0
$M^{(2,2)}$	6	2	2	0	0
$M^{(2,1,1)}$	12	2	0	0	0
$M^{(1,1,1,1)}$	24	0	0	0	0

On comparing this table with the one that we obtained in the previous chapter we see that these M^λ's are not irreducible. So, our next task would be to construct the irreducible representations of S_n.

3.2 Specht Modules

In the previous section, we constructed a representation of S_n which was not necessarily irreducible. In this section, our aim is to construct irreducible representations of S_n. We shall consider an irreducible representation of M^λ that will correspond uniquely to λ.

We observe that a tabloid is fixed by the permutations which only permutes the entries of the rows among themselves in the action of S_n on tabloids. This gives us the motivation to have the following definition.

Definition 3.2.1. For a tableau t of size n, the **row group** of t, denoted by R_t is the subgroup of S_n consisting of permutations which only permutes the elements within each row of t. Similarly we can define the **column group** of t, denoted by C_t to be the subgroup of S_n which consists of permutations that only permute the elements within each column of t.

For instance, if

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

then $R_t = S_{\{1,2\}} \times S_{\{3\}}$ and $C_t = S_{\{1,3\}} \times S_{\{2\}}$.

Definition 3.2.2. If t is a tableau, then the associated **polytabloid** is

$$e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi\{t\}.$$

To see that S_n acts on the set of polytabloids we have the following lemma.

Lemma 3.2.1. *Let t be a tableau and π be a permutation, then $e_{\pi t} = \pi e_t$.*

Proof. We first observe that $C_{\pi t} = \pi C_t \pi^{-1}$ and then we have

$$\begin{aligned} e_{\pi t} &= \sum_{\sigma \in C_{\pi t}} \mathrm{sgn}(\sigma)\sigma\{\pi t\} \\ &= \sum_{\sigma \in \pi C_t \pi^{-1}} \mathrm{sgn}(\sigma)\sigma\{\pi t\} \\ &= \sum_{\sigma' \in C_t} \mathrm{sgn}(\pi\sigma'\pi^{-1})\pi\sigma'\pi^{-1}\{\pi t\} \\ &= \pi \sum_{\sigma' \in C_t} \mathrm{sgn}(\sigma')\sigma'\{t\} \\ &= e_{\pi t}. \end{aligned} \tag{3.1}$$

□

Definition 3.2.3. The **Specht module** corresponding to a partition λ denoted by S^λ is the submodule of M^λ spanned by the polytabloids e_t, where t is taken over all tabloids of shape λ.

We look at some standard examples before discussing the properties of these objects and note that Lemma 3.2.1 shows that S^λ is a representation of S_n.

Example 3.2.4. Let $\lambda = (n)$. As before S^λ is the one dimensional trivial representation.

Example 3.2.5. Let $\lambda = (1^n) = (1, 1, \ldots, 1)$ and t be given by the following tabloid

1
2
.
.
.
n

By definition, e_t is the sum of all λ-tabloids multiplied by the sign of the permutation. For any other tabloid t', we have $e_t = \pm e_{t'}$, so we see that here S^λ is the alternating representation.

Example 3.2.6. Taking $\lambda = (n-1, 1)$, using the same notations as Example 3.1.9 we see that the polytabloids have the form $\{t_i\} - \{t_j\}$. Then, S^λ is spanned by elements of the form $E_i - E_j$ where E_i is the tabloid $\{t_i\}$. Thus we have

$$S^\lambda = \{c_1E_1 + c_2E_2 + \cdots + c_nE_n \mid c_1 + c_2 + \cdots + c_n = 0\}.$$

This is the *standard representation* that we found in the previous chapter. Also, $S^{(n-1,1)} \oplus S^{(n)} = M^{(n-1,1)}$.

From the above examples, we see that S^λs have described all the irreducible representations of S_3. This is in general also true.

Theorem 3.2.2. *The Specht modules S^λ forms a complete list of irreducible representations of S_n, where $\lambda \vdash n$.*

The proof of this result is in two parts which we sketch here. A detailed proof can be found in [2] or [3]. First, we have to show that S^λ is an irreducible sub-module of M^λ and then we show that $S^\lambda \neq S^\mu$ if $\lambda \neq \mu$. Since the number of conjugacy classes of S_n is equal to the number of partitions, so we will have a complete list of the irreducible representations of S_n.

To prove that the Specht modules are irreducible, we have three main things to do. First, we define an inner product on M^λ, then we do a projection operation like we defined the polytabloids above. Finally we use the following lemma to complete the proof.

Lemma 3.2.3 (Submodule Lemma)**.** *Let V be a sub-module of M^λ, then either $S^\lambda \subset V$ or $V \subset (S^\lambda)^\perp$.*

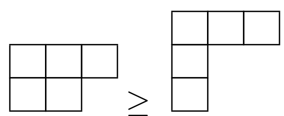
To prove that the Specht modules are distinct we need the following definition.

Definition 3.2.7. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$, then we say that λ dominates μ, written as $\lambda \geq \mu$, if for each $1 \leq i \leq \max(l, r)$ we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i.$$

If $l > r$, we set $\mu_i = 0$ for $i > r$ and vice-versa.

For example,



This is a partial ordering on the set of partitions of n via the Young diagrams.

Lemma 3.2.4 (**Dominance Lemma**). *Let t and s be tableaux of shape λ and μ respectively. If for each i, the elements of row i of s are all in different columns of t, then $\lambda \geq \mu$.*

Proof. We sort the entries in each column of t so that the elements in the first i rows of s occurs in the first i rows of t. This is possible because the elements of row i of t are in different columns of s. And then we have the result. □

Theorem 3.2.5. *Let $\theta \in \mathrm{Hom}(S^\lambda, M^\mu)$ be a non-zero map of representations. Then $\lambda \geq \mu$.*

Corollary 3.2.6. *If $S^\lambda \simeq S^\mu$ then $\mu = \lambda$.*

Proof. We have $S^\mu \hookrightarrow M^\mu$ and $S^\lambda \hookrightarrow M^\lambda$. By the above theorem we have $\lambda \geq \mu$ and $\mu \geq \lambda$ and hence the result. □

Corollary 3.2.7. *The only irreducible representations in M^μ are S^λ for $\lambda \geq \mu$.*

3.3 RSK Correspondence

In general, the polytabloids that we defined earlier are not independent. In fact, any pair of polytabloids in $S^{(1^n)}$ are linearly dependent. So, our immediate task is to find a suitable basis for S^λ to have some knowledge about its dimension and character. The first step in this direction is the **Richardson-Schensted-Knuth** (RSK) correspondence. The results will be briefly sketched here. We follow the exposition in [2], [3] and [5], where a more rigorous treatment is available.

We denote by f^λ the number of standard tableau of shape λ. A tableau is called **semi-standard** if the entries are not necessarily distinct, but the rows are weakly increasing and columns are strictly increasing. The basic operation in the RSK algorithm is **row insertion** in a semi-standard tableau. To row insert the number m into row j of the tableau t we proceed as follows: If no number in row j exceeds m then we put m at the end of row j, else, we find the smallest number not less than m and replace it with m, choosing the leftmost if there are more than one such number. The number that we replaced will be called the **bumped out** number and we row insert it into row $j+1$ and continue the process over. If at any point we are in an empty row then we just create a new box in that row and insert our number. Such a row inserted tableau is denoted by $m \leftarrow t$.

Example 3.3.1. We shall row insert 2 into the semi-standard tableau

1	2	2	3	5
2	3	5	5	
4	4	7		

Here 2 bumps 3 out of the first row, 3 bumps the first 5 out of the second row, 5 then bumps the 7 out of the third row, which then goes into the final row to give us the following tableau.

1	2	2	2	5
2	3	3	5	
4	4	5		
7				

A reverse algorithm also exists to recover our original tableau if we are given the final tableau and the final box that was added to the tableau. We note that if t is standard and m does not lie in t then $m \leftarrow t$ is also standard and the same is true if t is semi-standard. The RSK algorithm is in fact a bijection between the $2 \times n$ lexicographically ordered arrays with entries from $[1, 2, \ldots, n]$, and pairs (s, t) of semi-standard tableau, where the entries of s are the numbers in the bottom row of the array and the entries of t are the numbers in the top row of the array. An array is ordered lexicographically if for $j \leq l$, whenever $\binom{i}{j}$ occurs before $\binom{k}{l}$ then either $i < k$ or $i = k$. Such an array is actually a permutation. We now briefly sketch the RSK algorithm for an array.

In the first step, we start with two empty tableaux, say s_0 and t_0. If the first entry in our two line array is $\binom{i}{j}$ then we row insert j into the first row of s. This creates a new box in s. We then find the corresponding box in t and put i in that box. This gives us two new tableaux, say s_1 and t_1.

Then, we repeat the first step for each pair in the array and end up with two tableaux, $s_n = s$ and $t_n = t$. Here s will be semi-standard because we are just performing row insertions. If i' is inserted in a box below k in t, then l will bump an entry from a row above. So, k is smaller than some previous entry in the array. As we are in lexicographic order, this happens if $k > i$ and hence, t is also semi-standard.

For example, if we have the following array

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 3 \\ 3 & 3 & 2 & 1 & 2 & 4 \end{pmatrix}$$

we get the following two semi-standard tableau at the end of our process.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

This algorithm is also reversible: given a pair of semi-standard tableaux (s, t) we can get the corresponding two-row array. However, we skip the proof, which can be found in [2].

Theorem 3.3.1. *The RSK correspondence is a bijection between lexicographically ordered $(2 \times n)$ arrays and pairs of semi-standard tableaux (s, t) with shape λ for some partition of n. If we restrict the arrays to permutations, then the RSK correspondence gives a bijection between elements of S_n and pairs of standard tableaux with entries from $[1, 2, \ldots, n]$.*

Corollary 3.3.2. *We have*

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!.$$

Theorem 3.3.3. *The standard polytabloids form a basis of S^λ.*

We only sketch the proof here, which can be found in [2] or [3]. First, we impose a partial ordering on the tabloids and then show that a greatest element in this ordering cannot be cancelled by adding a linear combination of other tabloids. Then using induction, the linear independence can be proved.

Corollary 3.3.4. *Let $\lambda \vdash n$, then $\dim S^\lambda = f^\lambda$.*

Proof. From Corollary 1.2.12 we have that if the irreducible representations of a group G of order g are $V_1, \ldots, V_r$ then

$$g = \sum_{i=1}^{r} \dim(V_i)^2.$$

We also know that each Specht module has dimension at least f^λ since it contains f^λ linearly independent polytabloids. We have also seen in this chapter that two Specht modules for different partitions are not isomorphic. So,

$$n! \geq \sum_{\lambda \vdash n} (\dim S^\lambda)^2 \geq \sum_{\lambda \vdash n} (f^\lambda)^2 = n!.$$

Thus, there is equality everywhere and we get our desired result. □

We note that still we do not have an explicit formula for the dimension of the Specht module. This is derived in the following sections.

3.4 Symmetric Functions

In this section, we study the symmetric functions briefly and relate the characteristic zero representations of S_n to the theory of symmetric polynomials. This section facilitates the next section where we derive a formula for the character of S^λ.

The ring of symmetric polynomials is constructed in m variables as a ring of fixed points in $\mathbb{Z}(x_1, x_2, \ldots, x_n)$ under the action of S_m. For any $\sigma \in S_m$, it acts on the monomials in this ring by $x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} \sigma = x_{1\sigma}^{a_1} x_{2\sigma}^{a_2} \cdots x_{m\sigma}^{a_m}$. For convenience, if $\lambda \vdash n$ we write $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m}$ for $m \geq k$, where k is the number of rows of λ. If t is a semi-standard λ-tableau we write $x^t = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ where a_i's are the number of times i occurs in t provided $n \geq m$. If μ is a partition of n with a_i parts of size i, then we denote $z(\mu) = 1^{a_1} a_1! 2^{a_2} a_2! \ldots n^{a_n} a_n!$. An easy exercise in algebra is to show that the number of conjugacy classes of S_n is given by $z(\mu)/n!$.

Definition 3.4.1. A **symmetric polynomial** is a polynomial in n variables such that the polynomial remains unchanged if we change the order of the variables.

Definition 3.4.2. A **symmetric function of degree** n is a collection of symmetric polynomials, $f^i(x_1, x_2, \ldots, x_i)$ for $i \geq k$, for some fixed k. If $j > i \geq k$ then $f^j(x_1, \ldots, x_i, 0, \ldots, 0) = f^i(x_1, \ldots, x_i)$. We denote the collection of all symmetric functions of degree n by Λ_n.

When the context is clear, we will denote the symmetric functions by just f and drop the superscripts and variables. We add and multiply symmetric functions by adding and multiplying the associated symmetric polynomials. Multiplication is a bilinear map $\Lambda_p \times \Lambda_q \longrightarrow \Lambda_{p+q}$ and with these conditions we see that $\Lambda = \oplus_{n\geq 0}\Lambda_n$, is the ring of all symmetric functions.

The next natural step is to look for a basis of Λ, but for that we shall need some more definitions.

Definition 3.4.3. In the following we have $\lambda \vdash n$ and $q \in \mathbb{N}$.

(i) The **monomial symmetric polynomial** $m_\lambda(x_1, x_2, \ldots, x_m)$ is the sum of all monomials arising from $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ by the action of S_m

$$m_\lambda(x) = \sum_{\sigma \in S_m} x^\lambda \sigma.$$

(ii) The **complete symmetric polynomial** $h_q(x_1, \ldots, x_m)$ is the sum of all monomials arising from products of $x_1, x_2, \ldots, x_m$ such that the total degree of each monomial is q. So,

$$h_q(x) = \sum_{\mu \vdash q} m_\mu(x).$$

(iii) The **elementary symmetric polynomial** $e_q(x_1, x_2, \ldots, x_m)$ is the sum of all monomials arising from products of $x_1, \ldots, x_m$ of total degree q where each x_i is used at most once. So, if $m < q$, then e_q vanishes.

(iv) The **power sum symmetric polynomial** $p_q(x_1, \ldots, x_m)$ is the sum of the qth powers of the variables. So,

$$p_q(x_1, \ldots, x_m) = \sum_{i=1}^{m} x_i^q.$$

(v) The **Schur polynomial** s_λ is

$$s_\lambda(x_1, \ldots, x_m) = \sum x^t,$$

where the sum is over all semi-standard λ-tableau t with entries in $[1, 2, \ldots, m]$. If $m < k$, then s_λ vanishes.

If $f(x_1, x_2, \ldots, x_m)$ is a symmetric polynomial and the monomial x^λ occurs in f then m_λ occurs in f and hence the m_λs forms a basis for Λ_n. So, $\dim \Lambda_n = p(n)$ (the number of partitions of n). For convenience, we denote $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_k}$, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots e_{\lambda_k}$ and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_k}$

Example 3.4.4. (i) $m_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1^2 x_3^2 + x_2^2 x_3 + x_2 x_3^2$.

(ii) $h_{(2,1)}(x_1, x_2) = (x_1^2 + x_2^2 + x_1 x_2)(x_1 + x_2)$.

(iii) $e_{(2,1)}(x_1, x_2) = x_1 x_2 (x_1 + x_2)$.

Theorem 3.4.1. *The Schur polynomial, s_λ is symmetric.*

Since S_m is generated by transpositions, so it is sufficient for us to prove that $s_\lambda(x_1, x_2, \ldots, x_m)(i\ i+1) = s_\lambda(x_1, x_2, \ldots, x_m)$ for each $1 \leq i \leq m-1$.

From [2] we know that the generating functions for $e_n(x)$ and $h_n(x)$ are as follows:

$$\sum_{n \geq 0} h_n(x) t^n = \prod_{i=1}^{m} \frac{1}{1 - x_i t}$$

and

$$\sum_{n=0}^{m} (-1)^n e_n(x) t^n = \prod_{i=1}^{m} (1 - x_i t).$$

Multiplying these together and finding the coefficients of t^n will give us

$$\sum_{i=1}^{n} (-1)^{n-i} h_i(x) e_{n-i}(x) = 0$$

which shows that $\langle h_i(x)\rangle = \langle e_i(x)\rangle$ and hence $(h_\lambda : \lambda \vdash n)$ is also a basis for Λ_n.

Following some more algebraic manipulations and comparing coefficients we get the following identity

$$h_n(x) = \sum_{\mu \vdash n} \frac{1}{z(\mu)} p_\mu(x).$$

So, $\Lambda_n = \langle p_\mu : \mu \vdash n\rangle_{\mathbb{Q}}$. Finally, we have that the Schur polynomials also give a basis for Λ_n, the proof of which depends on the following result which we shall not prove here.

Theorem 3.4.2 (Pieri's Rule)**.** *Let $\lambda \vdash n$ and $q \in \mathbb{N}$, then we have*

$$s_\lambda(x).h_q(x) = \sum_\mu s_\mu(x),$$

where the sum is taken over all $\mu \vdash n+q$ which we obtain by adding q boxes into λ, no two in the same column.

We now prove that the Schur functions form a basis for Λ_n. We have $h_q(x) = s_{(q)}(x)$. Expanding $h_\lambda(x)$ as a sum of Schur polynomials by considering the h_{λ_1} as $s_{(\lambda_1)}$ and then using Pieri's Rule to multiply it with h_{λ_2} and repeating the process until we can write it in the form

$$h_\mu(x) = \sum_\lambda K_{\lambda\mu} s_\lambda(x).$$

Here the coefficients are called **Kostka numbers**, which is the number of semi-standard λ-tabloids t with μ_i i's. We then say that λ is of the type μ. If $K_{\lambda\mu} \neq 0$ then λ dominates μ. We now introduce an inner product on Λ defined by an orthonormal basis formed by the Schur functions s_λ.

Theorem 3.4.3. *Let $\lambda, \mu \vdash n$. Then we have the following:*

(i) $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$,

(ii) $\langle m_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$,

(iii) $\langle p_\lambda, p_\mu \rangle = \dfrac{\delta_{\lambda\mu}}{z(\lambda)}$.

Definition 3.4.5. Let $\lambda, \mu \vdash n$, we define the coefficients $\xi_{\lambda\mu}$ and $\chi_{\lambda\mu}$ by

$$p_\mu = \sum_{\lambda \vdash n} \xi_{\lambda\mu} m_\lambda$$

and

$$p_\mu = \sum_{\lambda \vdash n} \chi_{\lambda\mu} s_l b.$$

Using Theorem 3.4.3 we can also derive the following equivalent identities

$$h_\lambda = \sum_{\mu \vdash n} \frac{\xi_{\lambda\mu}}{z(\mu)} p_\mu$$

and

$$s_\lambda = \sum_{\mu \vdash n} \frac{\chi_{\lambda\mu}}{z(\mu)} p_\mu.$$

3.5 Frobenius Character Formula

The aim of this section is to enunciate the character of the Specht modules and to state an important result in combinatorics, called the **hook length formula**. We also go through the induced and restricted representations of S_n.

Theorem 3.5.1 (Frobenius Character Formula)**.** *Let $\lambda \vdash n$ with k rows and also let $l_i = \lambda_i + k - i$. Then the character of S^λ on the conjugacy class of a partition μ, denoted by $\chi_{\lambda\mu}$ is the coefficient of $x_1^{l_1} x_2^{l_2} \dots x_k^{l_k}$ in the polynomial*

$$\prod_{1 \le i < j \le k} (x_i - x_j) . p_\mu(x_1, x_2, \dots, x_k).$$

This theorem is a consequence of the following theorem, which we state without proof.

Theorem 3.5.2 (Jacobi-Trudi Formula)**.** *Let $\lambda \vdash n$ with k rows. We define $\Delta_\lambda(x)$ to be the determinant of the $k \times k$ matrix A, where $A_{ij} = x_i^{\lambda_j + k - j}$. We also let $\Delta_0(x)$ to be the Vandermonde determinant* $\det(x_i^{j-1}) = \prod_{i<j}(x_i - x_j)$. *Then we have*

$$s_\lambda(x_1, \ldots, x_k) = \frac{\Delta_\lambda(x)}{\Delta_0(x)}.$$

A proof of this result can be found in [5].

We now find the dimension of S^λ by evaluating its character on the identity. Thus, by Theorem 3.5.1, it is the coefficient of x^l in

$$\prod_{1 \le i < j \le n} (x_i - x_j)(x_1 + \cdots + x_k)^n.$$

After expanding the Vandermonde deteminant and using the multinomial theorem in the other expansion along with some algebraic manipulations, we get

$$\dim S^\lambda = \frac{n!}{\prod_{i=1}^k l_i! \prod_{1 \le i < j \le k}(l_i - l_j)}.$$

We now introduce the concept of a **hook** in a partition λ. The hook on a given node is the number of boxes to the right as well as below that node plus counting the node we are in, once. The **hook-length** of a hook is the number of boxes that the hook involves. For example in the Young diagram below, we mark each box with its hook-length.

6	4	2	1
3	1		
1			

The l_i's that we defined earlier are just the hook-length of the nodes in the first column of λ. If α is a node on λ, then we denote its hook-length by h_α.

d

Proof. We have to show

$$\frac{n!}{\prod_{i=1}^{k} l_i! \prod_{1\le i<j\le k}(l_i - l_j)} = \frac{n!}{\prod_\alpha h_\alpha}.$$

We shall use induction on the columns of λ. If λ has only one column, then $\lambda = (1^n)$ and we have

$$\dim S^\lambda = \frac{n!}{n!(n-1)!\cdots 1!}\prod_{i=1}^{n-1}(n-i)! = 1$$

which is the same as the hook-length formula.

Let λ' be the partition of $n-k$ that is obtained after removing the first column of λ. Let k' be the number of rows of λ', and let l'_i be the hook length of the ith node in the first column of λ'. Then, we have $l'_i - l'_{i+1} = l_i - l_{i+1}$ and $l'_i = l_i - (k-k') - 1$. Now, using the induction hypothesis we have,

$$\begin{aligned}\dim S^\lambda &= \frac{n!}{\prod_{i=1}^{k} l}\frac{\prod(l_i - l_j)}{\prod_{i=1}^{k}(l_i-1)!} \qquad (3.2)\\ &= \frac{n!}{\prod_{\alpha\in\lambda} h_\alpha}\frac{\prod_{i=1}^{k} l'_i!}{\prod_{i=1}^{k}(l_i-1)!}\prod_{1\le i<j\le k, j>k}(l_i - l_j).\end{aligned}$$

It remains to show that the last terms in the product cancel out to get the desired result. We have, if $i > k', l_i = k - i + 1$, then

$$\begin{aligned}\prod_{1\le i<j\le k, j>k'}(l_i - l_j) &= \prod_{i=1}^{k}\prod_{j>\max(i,k')}(l_i - l_j) \qquad (3.3)\\ &= \prod_{i=1}^{k-1}\prod_{j>\max(i,k')}\frac{(l_i-1)!}{(l_i - 1 + k + \max(i,k'))!}\\ &= \prod_{i=1}^{k'}\frac{(l_i-1)!}{l'_i!}\prod_{i=k'+1}^{k}(l_i-1)!\end{aligned}$$

The hook length formula follows from this. □

We now come to the final topic of our thesis: the induced and restricted representations of S_n to S_{n+1} and S_{n-1} respectively. We will only mention the results here, without proofs. More details can be found in [2] or [3].

Much like partitions, Young diagrams can also be partially ordered by inclusion. The resulting partially ordered set is called a **Young's lattice**. Let $\lambda \nearrow \mu$ denote that μ can be obtained from λ by adding a box to λ to get a new partition. Continuing this way from the empty Young digram we can move upwards adding one box at a time, and get all the partitions of n at the nth level of this lattice.

Theorem 3.5.3 (Branching Rule)**.** *Let $\lambda \vdash n$, then we have*

$$\mathrm{Res}_{S_{n-1}}^{S_n} S^\lambda \cong \oplus_{\mu:\mu\nearrow\lambda} S^\mu$$

and

$$\mathrm{Ind}_{S_n}^{S_{n+1}} S^\lambda \cong \oplus_{\mu:\lambda\nearrow\mu} S^\mu.$$

The proof of this result uses the Frobenius Reciprocity Theorem, that we derived in the first chapter. Before proceeding further, we look at an example. Let us look at the trivial representation of the trivial group, which corresponds to the partition (1) of S_1. Inducing to S_2 we have

$$\mathrm{Ind}_{S_1}^{S_2} S^{(1)} = S^{(1,1)} \oplus S^{(2)}.$$

Again inducing it to S_3 and then finally to S_4 we shall get

$$\mathrm{Ind}_{S_1}^{S_4} S^{(1)} = S^{(4)} \oplus (S^{(3,1)})^3 \oplus (S^{(2,1,1)})^3 \oplus (S^{(2,2)})^2 \oplus S^{(1,1,1,1)}.$$

So far, we have constructed M^λ and from it we found irreducible representations S^λ which forms a complete list of such representations for S_n as λ varies over all partitions of n. But we have not yet discussed about the decomposition of these M^λ. We have the following results in this direction.

Theorem 3.5.4. *M^μ contains S^λ as a subrepresentation if and only if $\lambda \geq \mu$. Also, M^μ contains only one copy of S^μ.*

Now, we are interested in finding the number of copies of S^λ in M^μ, and for this we must make use of the Kostka numbers. One basic property of these numbers is that $K_{\lambda\lambda} = 1$. This is clear because there can be only one semi-standard tableau of the partition λ of type λ. We now have the final result on decomposition of the M^μ's.

Theorem 3.5.5 (Young's Rule)**.**

$$M^\mu \cong \oplus_{\lambda \geq \mu} K_{\lambda\mu} S^\lambda.$$

We close this chapter with the following examples.

Example 3.5.1. We have $K_{(n)\mu} = 1$ as there is only one possible (n)-semistandard tableau of type μ. Young's rule then says that every M^μ contains exactly one copy of the trivial representation $S^{(n)}$, which we have already seen in a previous example.

Example 3.5.2. We note that $K_{\lambda(1^n)} = f^\lambda$, so by Young's Rule we have $M^{(1^n)} \cong \oplus_\lambda f^\lambda S^\lambda$.

By taking the magnitude of the characters in the above, we get another proof of the fact that $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$.

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