A SCORE AND A DOZEN

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ABSTRACT. The aim of this article is to present a few solutions to problems based on the number 2012 and show the importance of the year's number for any olympiad.

1. INTRODUCTION

Hello and thanks for showing your interest towards mathematics by opening this article. To understand it completely you will require a bit of olympiad knowledge but if someone finds it boring then I can't help it. If you you have ever experienced any olympiad then you may be knowing through your personal experience that every year in various olympiads the year in question plays a great role. Almost in all olympiads at least one question depends on the year number(This year its 2012 :-D).

From my own personal experience I have seen that in the INMO (Indian National Mathematical Olympiad) rarely do such questions come. But they do come in the RMO (Regional Mathematical Olympiads) and ofcourse in the prestigious IMO(International Mathematical Olympiad). I still remember when last year while preparing for the olympiads I was factorising 2012, which is, $2^2 \cdot 503$(Its obvious that such a trivial thing would never come but it saves time in case it is necessary to factorise). In the IMO there are 6 questions to be solved in a span of nine hours in two days. Out of those 6 questions about $16 - 33$ percent questions might contain the use of the year. This has been used already in certain competions such as Stanford Math Tournament, American Invitational Mathematics Examination, and definitely will be used in the others yet to come. The list is non-exhaustive and a lot of original ideas can combine to form many more. So lets bring it on and take the test. I would request you to please try these sums[1] on your own before seeing the solutions.

[1]Some of these sums are a collection from various olympiads. The others have been contributed by Prof. M.B. Rege, and members of the Art of Problem Solving http://www.mathlinks.ro forum. Some of them have been created by the author(*) also.

2. The Problem set

(1) What is the unit's digit of $2012^{2012^{2012^{2012\ldots 2012 \text{ times}}}}$? (*)

(2) Find a compact form for $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + 2012.2012!$. (Canadian Mathematical Olympiad 1969)

(3) Show that $\exists$ inifitely many n such that $n \in \mathbb{N}$ and $n \mid 2012^n + 1$. (*)

(4) Let $A = 1! \cdot 2! \cdot 3! \ldots 2012!$. Is it true that $\exists$ some x such that $1 \leq x \leq 2012$ and $x \in \mathbb{N}$ and $\dfrac{A}{x!}$ is a perfect square ? (Prof. Mangesh Rege, NEHU)

(5) If $1! + 2! + 3! + \ldots + 2012! = a^b$, for some $a \in \mathbb{N}$ and $b \in \mathbb{N}$,then b can be atmost 1. (Prof. Mangesh Rege, NEHU)

(6) For what value of $x (x \geq 3$ and $x \in \mathbb{N})$ is the number $(2012)_x$ when expressed in decimal base divisible by 12 ? (*)

(7) Find all integral solutions for a, b and c for the equation :–

$$a^2 + b^2 + c^2 = 2012abc$$

(*)

(8) Find the number of positive integral solutions to the equation :–

$$a^2 + 2012^2 = b^2$$

(*)

(9) 2012 balls are put into 1006 boxes in such a way that none of the boxes is empty. Is it the case that certain boxes together always make 1006 balls ? (Mangesh B. Rege, NEHU)

(10) An n-dice is a dice with n surfaces. Let there be a 2012-dice whose opposite sides have the numbers paired up as $\{1, 2\}, \{3, 4\}, \ldots, \{2011, 2012\}$. Let the pairs be called $A_1, A_2, \ldots, A_{1006}$. For any $1 \leq a \leq 2012$, $f(a)$ is the element contained in the set $A_{[\frac{a+1}{2}]} - \{a\}$. The dice is rolled 2012 times and the readings are noted as $b_1, b_2, \ldots, b_{2012}$. $A = \Pi_{i=1}^{2012} b_i$, $B = (\Pi_{i=1}^{1006}(2012 - b_{2i-1}))(\Pi_{i=1}^{1006}(2012 - f(b_{2i})))$. If b_i does not contain any of the elements of A_i's more than twice then prove that $3 \mid AB$. (*)

(11) $X_1 = 9$, $X_n = 3X_{n-1}^4 + 4X_{n-1}^3$. X_2 ends in 99. Prove that X_{12} ends in atleast 2012 9's. (Mangesh B. Rege)

(12) Find the value of n such that :–
- (i) n is of the form $2012^k + 1$ for some $k \in \mathbb{N}$

- (ii) 2012 is a root of the equation

$$2012x^{2012} - 2011x^{2011} + \ldots - x + n = 0$$

(*)

(13) Find the number of positive integral solutions to the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2012}$$

(By mathbuzz, member of AOPS)

(14) Is it possible to make 2012 using the numbers $1^2 \square 2^2 \square \ldots \square n^n$ for some n natural numbers and using only the symbols $+, -$ in the blank spaces? (By mathbuzz, member of AOPS)

(15) Let $d(n)$ be the digital sum of $n \in \mathbb{N}$. Solve $n + d(n) + d(d(n)) + \ldots + d^{2009}(n) = 2012^{2012}$ ($d^{2012}(n)$ is the digital sum composed over n 2010 times. (By Faustus, member of AOPS)

(16) Find the number of trailing zeroes in 2012! and also find the last non-zero digit in it.(*)

(17) Into each box of a 2012×2012 square grid, a real number greater than or equal to 0 and less than equal to 1 is inserted. Consider splitting the grid into two non-empty rectangles consisting of boxes of the grid by drawing a line parallel either to the horizontal or vertical side of the grid. Suppose that for at one of the resulting rectangles the sum of the numbers in the boxes within the rectangle is less than or equal to 1, no matter how the grid is split into two such rectangles.Determine the maximum possible value for the sum of all the 2012×2012 numbers inserted into the boxes. (APMO 2012)

(18) Let $S = \{1, 2, \ldots, 2012\}$.We want to partition S into two disjoint sets such that both sets do not contain two different numbers whose sum is a power of 2. Find the number of such partitions.(Turkey JBMO TST 2012)

(19) Find the number of ordered pairs of positive integer solutions (m, n) to the equation

$$20m + 12n = 2012$$

(USA AIME, 2012)

(20) Find the number of ordered triples (A, B, C) such that $A \cup B \cup C = \{1, 2, \ldots, 2012\}$ and $A \cap B \cap C = \phi$

(21) In some squares of a 2012×2012 grid there are some beetles, such that no square contain more than one beetle. At one moment, all the beetles fly off the grid and then land on the grid again, also satisfying the condition that there is at most one beetle standing in each square. The vector from the centre of the square from which a beetle B flies to the centre of the square on which it lands is called the translation vector of beetle B. For all possible starting and ending configurations, find the maximum length of the sum of the translation vectors of all beetles. (China TST, 2012)

(22) Let x, y be real numbers such that $x > 2010$ and $y > 2011$ satisfy $2010\sqrt{(x+2010)(x-2010)}+2011\sqrt{(y+2011)(y-2011)} = \frac{1}{2}(x^2+y^2)$. Compute $x+y$.

(23) Find the sum of all integers x, $x \geq 3$, such that 201020112012_x (that is 201020112012 intepreted as a base x number) is divisible by $x-1$. (USA Stanford Mathematics Tournament)

(24) The expression $\circ 1 \circ 2 \circ 3 \circ \ldots \circ 2012$ is written on a blackboard. Catherine places a $+$ sign or a $-$ sign into each blank. She then eveluates the expression, and finds the remainder when it is divided by 2012. How many possible values are there for this remainder? (USA NIMO 2012)

(25) A convex 2012–gon $A_1A_2A_3\ldots A_{2012}$ has the property that for every integer $1 \leq i \leq 1006$, A_iA_{i+1006} partitions the polygon into two congruent regions. Show that for every pair of integers $1 \leq j < k \leq 1006$, quadrilateral $A_jA_kA_{j+1006}A_{k+1006}$ is parallelogram. (USA NIMO 2012)

3. Answers

Before seeing this we should give atleast $15-45$ minutes to each problem and then check this out. After solving any of the problems we should not think that we are finished with the problem. We must try to search for other solutions, find relations between different solutions and try to find generalisations.

We do not provide the solutions to all of these problems but leave them to the reader. At certain points we provide only mere hints to the problems.

(1) 6

(2) Use the fact that $i \cdot i! = (i+1)! - i!$. Or find a pattern and use induction.

(3) Notice that $2013 \mid 2012^{2013}+1$. Using induction generalize the above. We leave the generalization to the reader.

(4) Write $(2x-1)! \cdot (2x)! = (2x-1)!^2 \cdot (2x)$. Then multiply all the terms and proceed. Required $x = 1006$.

(5) It is trivial that b cannot be even(Check the unit's digit of the number). If $b \geq 3$ then for any prime p dividing the number p^3 should also divide the number. Notice that 3 divides the number but 27 does not.

(6) There is no value for x.

(7) Use infinite decent. The only solution is $(a, b, c) = (0, 0, 0)$.

(8) Quite easy. Work it out into cases.

(9) Use pigeonhole principle.

(10) Here we use proof by contradiction. First it is noticeable that none of the b_i's are congruent $0, 1 \bmod 3$. Hence all of them are congruent $1 \bmod 3$ which is not possible since a number does not repeat more than twice.

(11) Challenge. Try it. Quite tough

(12) Take the LHS of the equation as $f(x)$. Notice that $f(0) = n$, odd. Hence if α is a root of the equation then $\alpha - 0 = \alpha \mid f(0) \Rightarrow \alpha \mid n$. If 2012 is a root of the equation then it has to divide n which is odd. Hence the answer would be no.

(13) Notice that $\frac{1}{x}+\frac{1}{y} = \frac{1}{2012} \Rightarrow x+y = \frac{xy}{2012} = \frac{(x-2012)(y-2012)+2012(x+y)-2012^2}{2012} = \frac{(x-2012)(y-2012)}{2012} + (x+y) - 2012$. Now proceed.

(14) Notice that $n - (n+1)^2 - (n+2)^2 + (n+3)^2 = 4$. Now proceed.

(15) Check $\pmod 9$ both sides.

(16) Use de Polignac's formula for number of trailing zeroes. Rest try. Answer: 501 and 6

(17) Answer: 5. First notice that 5 is possible as we can put 1 in the boxes $(2,1), (2,2), (2,3), (1,2), (3,2)$ and 0 in all the other boxes. Now to show that it is the maximum possible sum. Suppose that the maximum attainable sum be S. Let the sum of all numbers in the m^{th} column be C_m. Hence, $S = \sum_{i=1}^{2012} C_i$. Now as soon as $C_1 + C_2 + \ldots + C_k > 1$, then $S - (C_1 + C_2 + \ldots + C_k) \leq 1$. Hence $C_1 + C_2 + \ldots + C_k \geq S - 1$. Among these first k columns if any of them have sum less than $S-2$ than it would be at most $S-3$. If the column has sum, $S-3$ then $S-3 = 1$ or else it would be able to partition the grid into two rectangles such that both sums strictly greater than 1. This yields $S = 4$ but we already know that for sum 5 also it is possible. Hence we neglect the case. By similar arguments we have that in that column there must be a number atleast $S-4$. Hence, $S - 4 \leq 1$. This yields $S = 5$.

(18) Answer:2^{10}. The highest power of 2 less than or equal to 2012 is 2^{10}. Hence the numbers from $2^{10}+1$ to 2012 will be determined automatically once the numbers from 1 to 2^{10} have been placed. Hence, we need worry about them. We require the following lemma:

Lemma: If the numbers $2^0, 2^1, 2^2, \dots, 2^{n-1}$ are placed then the numbers $1, 3, 5, 6, 7, \dots, 2^n - 1$ are automaticallly placed.

Proof. We use induction. If we determine where to put the numbers $1, 2, 4$ then we would place 3 in the set where 1 has been placed. Let us assume the lemma to be true for some particular k for some $k \in \mathbb{N}$. Hence we have to show that it is true for $k+1$. Let us place the numbers $2^0, 2^1, \dots, 2^k \Rightarrow$ we place the numbers $1, 2, 3, \dots, 2^{k+1} - 1$. Let us then place the number 2^{k+1}. Using the identity $(2^{k+1} - l) + (2^{k+1} + l) = 2^{k+2}$ we get that the numbers $2^{k+1} + 1, 2^{k+1} + 2, \dots, 2^{k+2} - 2$ are automatically determined. $\square$

Hence the total number of partitions is equivalent to determining the number of partitions of $2^0, 2^1, 2^2, \dots, 2^{10}$ which is equal to 2^{10}

(19) Divide both sides of the equation by 4 and use the general form for solving a linear diophantine equation in two variables.

(20) Answer:6^{2012}. Each of the numbers have to be in at least one of the sets but none of them can be in all the three sets. Therefore for each element of the set $1, 2, 3, \dots, 2012$ for plugging into the sets we have the following options: $A, B, C, (A, B), (B, C), (C, A)$. Hence we have 6^{2012} possibilities.

(21) Answer: $\dfrac{1}{4}2012^3$. Say beetles A and B are such that A moves to where B started, and B moves somewhere else. We can replace A and B with beetle C that travels from the starting location of A to the ending of B. Thus the maximum length of the sum of the translation vectors can be achieved in the case where no A goes where B starts; i.e. no square is both a starting location and an ending location.

Thus it boils down to matching half of the squares with the other half to produce $\dfrac{1}{2} \times 2012^2$ vectors, the first half being the heads and the second the tails of the vectors. But knowing how vectors add it is irrelevant how they are matched; we need only pick which ones are heads and which are tails, then add the vectors from origin to the heads and subtract the vectors from origin to the tails. Note that since we can "reverse" any vector in the sum we will always want to use up all of the squares and not leave some of them empty before and after.

A final simplification is that, supposing the final vector sum is along direction V, it suffices to choose the half of the squares with positive dot product with V to be heads; and the other half to be tails. Thus the direction V of the final vector sum determines how the heads and tails are chosen to get a maximal sum.

The maximum occurs when V is along one of the coordinate axes. This is equivalent to having each square in the left half of the square contain a beetle at the beginning, and then having every beetle move 1006 squares to the right.

(22) Answer:$4023\sqrt{2}$. Using AM-GM we have $2010\sqrt{(x-2010)(x+2010)} = 2010\sqrt{x^2 - 2010^2} \leq \frac{2010^2 + x^2 - 2012}{2} = \frac{x^2}{2}$. Similarly for y.

(23) Writing the number in decimal base gives us $12 \equiv 0 \bmod x-1$. This gives the answer as 32.

(24) Answer: 1006. For the remainder we have to go $\pmod{2012}$. If all have $+$ signs then the remainder is 1006. If the sum of the numbers having a $-$ sign before them is denoted by S then the overall expression would sum out to be $1006(2011) - 2S \equiv 1006 - 2S \bmod 2012$. This gives that the remainders must be even. If only 1 number has a $-$ sign preceeding it and the rest have $+$ signs then the remainder would be $1006 - 2n$ where $n \in \mathbb{N}, 1 < n < 2012$. Hence all the even remainders are covered.

(25) It can be shown that the opposite pairs of the quadrilateral are equal in length. Hence the quadrilateral is a parallelogram

4. Acknowledgement

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5. Conclusion

I am looking forward to your suggestions. Other solutions to the problems are always acceptable. I would be happy if anyone is able to add more questions to this list. Thanks for your patience for going through this.

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