Pre Mathematics Olympiad Training
Women's College, Shillong

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I discussed the following problems during the two days of my class in the pre-INMO camp at Women's College Shillong. Thanks a lot to Rege Sir for inviting me to the camp. Hope to provide better and more tough problems in future !!

1 Tricks and fun

Ex 1.1 *A shopkeeper sells one chocolate for one rupee and also offers one chocolate in exchange for three wrappers. If you have 15 rupees with you, then how many chocolates can you buy from him?*

Ex 1.2 *Given the word STANDARD, take away two letters and add three digits to make a logical sequence.*

Ex 1.3 *Cross out 10 digits from the number 1234512345123451234512345 so that the remaining number is as large as possible.*

Ex 1.4 *What is the maximum sum of the number of Saturdays and Sundays in a leap year?*

Ex 1.5 *In a certain year there were exactly four Mondays and four Fridays in January. On which day did Republic Day fall on that year?*

Ex 1.6 *In a year N, the 300th day is Tuesday. In year N+1, the 200th day is a Tuesday. On which day of the week did the 100th day of the year N-1 occur?*

Ex 1.7 *You have a* $10cm \times 10cm$ *square piece of paper. How can you fold or reshape it to make a square of area* $50cm^2$ *?*

Ex 1.8 *You have a 10 cm by 10 cm cube which is made up of little 1 cm by 1 cm cubes. You paint the outside of the big cube red. How many of the little cubes get painted?*

Ex 1.9 *A number of children are standing around a circular table. They are evenly spaced and the 7th child is exactly opposite to the 18th child. How many children are there?*

Ex 1.10 *A 25 ft ladder is placed with its foot 7 ft away from a building. If the top of the ladder slips down 4 ft, how many feet will the bottom slide out?*

Ex 1.11 *Solve :* $|x-3|+|x+1|=4$

Ex 1.12 *Find the remainder when* $(m+1)^4$ *is divided by* m. *Hence find the remainder when* 17^{21} *is divided by 16.*

Ex 1.13 *A number consists of 300 digits out of which there are one hundred 2's, one hundred 3's and one hundred 7's (in whatever order possible). Whether this number is a perfect square?*

Ex 1.14 *A number consists of 300 digits out of which there are one hundred 0's, one hundred 1's and one hundred 2's (in whatever order possible). Whether this number is a perfect square?*

Ex 1.15 *The numbers from 1 to 10 are written in a row. Can you place* $+$ *and* $-$ *signs between them so that the sum is 0?*

Ex 1.16 *Find the number of zeros at the end of the product* $25 \times 35 \times 20 \times 22 \times 45 \times 75 \times 98 \times 65 \times 65 \times 125$.

Ex 1.17 *What least value must be assigned to * so that* $197 * 5462$ *is divisible by 9?*

Ex 1.18 *Find out all possible values of* A *and* H, *if* $A21H$ *is a four digit number divisible by 9.*

Ex 1.19 *Find digits* x *and* y *such that* $8x41y69$ *is divisible by 7.*

Ex 1.20 *Find the digit(s)* a *such that* $6a106$ *is divisible by 11.*

Ex 1.21 *Find the number of positive integers between 1 and 1000 which are divisible neither by 2 nor by 5.*

Ex 1.22 *A four digit number has the following properties:*

- *It is a perfect square.*
- *The first two digits are equal.*
- *The last two digits are equal.*

Find all such numbers.

Ex 1.23 *The numbers 147 and 225 after being divide by a two digit number leave the same remainder. Find the divisors.*

Ex 1.24 *There are two bells which rings after 20 and 30 seconds respectively. If they both ring together at 10 AM, then how many times together would they ring till 11 AM?*

Ex 1.25 *The product of two numbers is 19772 and their HCF is 16. Find all such possible pairs of numbers.*

Ex 1.26 *Four prime numbers are written in ascending order of magnitude. Product of the first three is 715 and that of the last three is 2431. What are the prime numbers?*

Ex 1.27 *If* $a_1, a_2, \ldots, a_{20}$ *are positive integers then show that*

$$(a_1 + a_2 + \ldots + a_{20})(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{20}}) \geq 400$$

Extend this result to n *positive integers.*

Ex 1.28 *If* $x, y, z > 0$ *and* $x + y + z = 1$, *then show that* $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9$.

Ex 1.29 *Can we draw a closed path made of 19 line segments each of which intersects exactly one of the other segments?*

Ex 1.30 *Can one make a change of Rs 25 using ten coins each of value Re 1 or Rs 5?*

Ex 1.31 *The product of 22 integers is equal to 1. Can their sum be zero?*

Ex 1.32 *Can one form a* 6×6 *magic square with the first 36 prime numbers? Magic square here means a* 6×6 *matrix with distinct entries such that the sum of the entries in each row, column and diagonal is a fixed number.*

Ex 1.33 *If* $x = 2$ *is a root of* $84x^3 - 157x^2 - kx + 78 = 0$*, find* k *and the other roots.*

Ex 1.34 *Suppose* a, b *are integers such that* $x^2 - x - 1$ *is a factor of* $ax^3 + bx^2 + 1$*. Find* a *and* b*.*

Ex 1.35 *Let* $a_1, a_2, \ldots,$ *be a sequence with the following properties:*

- $a_1 = 1$
- $a_{2n} = na_n$ *for any positive integer* n*.*

Find a_{2^n} *and hence find* $a_{2^{100}}$*.*

Ex 1.36 *Is* $512^3 + 675^3 + 720^3$ *a composite number? (Special thanks to Rege Sir for this question and for providing its beautiful solution!!)*

Ex 1.37 *Factorize : 99899 (Special thanks to Rege Sir for this question and for providing its beautiful solution!!)*

2 Geometry

Ex 2.1 *A triangle of unit area has sides* a, b, c *with* $a \geq b \geq c$*. Show that* $b \geq \sqrt{2}$*.*

Ex 2.2 *In a triangle ABC,* $\angle A = 90°$*. Median AM, angle bisector AK and altitude AH are drawn. Prove that* $\angle MAK = \angle KAH$*.*

Ex 2.3 *Prove that in a right angled triangle, the length of the median from the right angular vertex is half the hypotenuse.*

Ex 2.4 *Let ABC be an acute angled triangle and CD be the altitude through C. If* $AB = 8$ *units and* $CD = 6$ *units, find the distance between the mid points of AD and BC.*

Ex 2.5 *In a triangle ABC points D and E respectively divide the sides BC and CA in the ratio* $\dfrac{BD}{DC} = m$ *and* $\dfrac{AE}{EC} = n$*. The segments AD and BE intersect at a point X. Find the ratio* $\dfrac{AX}{XD}$*.*

Ex 2.6 *On the sides BC, CA and AB of a triandle ABC, points D,E,F are taken in such a way that* $\dfrac{BD}{DC} = \dfrac{CE}{EA} = \dfrac{AF}{FB} = 2$*. Show that the area of the triangle determined by the lines AD, BE, CF is* $\dfrac{\Delta}{7}$*, where* Δ *is the area of triangle ABC.*

Ex 2.7 *D, E, F are points on the sides BC, CA and AB respectively, of triangle ABC such that AD, BE and CF are concurrent at O. Show that*

1. $\dfrac{OD}{AD} + \dfrac{OE}{BE} + \dfrac{OF}{CF} = 1$
2. $\dfrac{AO}{AD} + \dfrac{BO}{BE} + \dfrac{CO}{CF} = 2$

3. $\dfrac{AO}{OD} = \dfrac{AF}{FB} + \dfrac{AE}{EC}$

Ex 2.8 *Medians BD and CE of ΔABC are perpendiculars. If BD=8 and CE=12, what is the area of ΔABC?*

Ex 2.9 *An acute isosceles triangle is inscribed in a circle. Tangents are drawn at B and C, meeting at point D with $\angle ABC = \angle ACB = 2\angle CDB$. What is the radian measure of $\angle BAC$?*

Ex 2.10 *Given line segments of length 1, a, b, construct line segments of lengths $a + b$, ab, $\dfrac{a}{b}$ and $\sqrt{a}$.*

Ex 2.11 *Given two points A and B construct a square of side AB.*

3 Number Theory

Ex 3.1 *Use the Division Algorithm to establish that (a) every odd integer is either of the form $4k - 1$ or $4k + 3$; (b) the square of any integer is either of the form $3k$ or $3k + 1$; (c) the cube of any integer is either of the form $9k$, $9k + 1$, or $9k + 8$.*

Ex 3.2 *Prove that no integer in the sequence $11, 111, 1111, 11111, \ldots$ is a perfect square.*

Ex 3.3 *Prove that, for any integer a, one of the integers a, $a + 2$, $a + 4$ is divisible by 3.*

Ex 3.4 *Prove that $4 \nmid (a^2 + 2)$for any integer a.*

Ex 3.5 *Verify that if an integer is simultaneously a square and a cube, then it must be either of the form $7k$ or $7k + 1$.*

Ex 3.6 *For $n \geq 1$, prove that $\dfrac{n(n+1)(2n+1)}{6}$ is an integer.*

Ex 3.7 *Show that there is no integer a such that $a^2 - 3a - 19$ is divisible by 289.*

Ex 3.8 *Show that there is no integer a such that $a^2 + a - 52$ is divisible by 121.*

Ex 3.9 *Show that there is no integer a such that $a^2 - 5a - 10$ is divisible by 169.*

Ex 3.10 *Find all integers n such that $n^2 + 1$ is divisible by $n + 1$.*

Ex 3.11 *What is the remainder when 2^{31} is divided by 5?*

Ex 3.12 *What is the unit digit of 777^{777}?*

Ex 3.13 *Find the remainder when 4333^3 is divided by 9.*

Ex 3.14 *Find the remainder when $13^{73} + 14^3$ is divided by 11.*

Ex 3.15 *Find the value of*

Ex 3.16 *Find all primes p such that both p and $p^2 + 8$ are primes.*

Ex 3.17 *Prove that for $n \geq 4$, $n, n + 2$ and $n + 4$ cannot be all primes.*

Ex 3.18 *Find all primes p such that p, $4p - 1$ and $8p - 1$ are all primes.*

Ex 3.19 *Prove that the fraction $\dfrac{21n + 4}{14n + 3}$ is irreducible for every natural number n.*

Ex 3.20 *Show that, given any n there exist n consecutive composite numbers.*

Ex 3.21 *From a book with pages numbered from 1 to 100 on both sides, some pages are torn off.The sum of the page numbers on the remaining pages is 4949. How many pages were torn off from the book?*

4 Combinatorics

Ex 4.1 *How many five digit numbers are there which are the same when the order of their digits is inverted?*

Ex 4.2 *Prove that the product of three consecutive natural numbers is always divisible by 6*

Ex 4.3 *Prove that the product of* r *consecutive natural numbers is always divisible by* $r!$.

Ex 4.4 *How many three digit numbers of the form* xyz*, where* $x < y$ *and* $z < y$ *can be formed?*

Ex 4.5 *Find the number of onto functions from* $A = \{a_1, a_2, \ldots, a_n\}$ *to* $B = \{b_1, b_2\}$

Ex 4.6 *Find the number of onto functions from* $A = \{a_1, a_2, \ldots, a_n\}$ *to* $B = \{b_1, b_2, b_3\}$

Ex 4.7 *There are five types of envelopes and four types of stamps in a post office. How many ways are there to buy a stamp and an envelope?*

Ex 4.8 *In how many ways 5 letters can be posted in 3 letter boxes?*

Ex 4.9 *Find the number of three-element subsets of* $\{1, 2, , \ldots, n\}$. *How many of these have the property that the sum of the elements is a multiple of 3?*

Ex 4.10 *In how many ways you can select an odd number of objects from* n *distinct objects?*

Ex 4.11 *What is the maximum number of points of intersection of all possible chords formed by* n *points on a circle?*

Ex 4.12 *Let* $X = \{0, 1, 2, \ldots, 10\}$. *Show that any subset* S *of* X *having 7 elements will contain two numbers whose sum is 10.*

Ex 4.13 *Fifty one points are scattered inside a square of side 1m. Prove that some set of 3 of these points can be covered by a square of side 20cm.*

Ex 4.14 *Suppose 101 integers are chosen at random from the integers* $1, 2, \ldots, 200$. *Show that among these integers there will be at least two such that one of them is divisible by the other.*