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Positive Definite Matrices

Introductory Article

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Abstract: In this article, we discuss about the positive definite matrices along with the semi positive definite matrices, which is defined as: *a n-by-n matrix A is said to be positive definite if* $x^*Ax > 0$, $\forall\ x(\neq 0) \in C^n$ *and similarly A is said to be positive semi-definite if* $x^*Ax \geq 0$, $\forall\ x(\neq 0) \in C^n$. *Here,* $x^* = x^T$, *when* $F = R$. We will study about its properties and characterizations.

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1. Introduction and Motivation

Before proceeding towards the main objective of the article, one need some preliminary concepts like *rank of a matrix, inner product, eigenvalues, eigenvectors, change of basis, similarity, diagonalization process, simultaneous diagonalization* etc. Those who donot have preliminary concepts on linear algebra, they may go through [2], [4], [5] and [6] and try to do some problem regarding these topics. Here, we assume that these concepts are clear to all. The proof of the results, we have stated here, can be found in either of [1], [2], [3], [4], [5] or [6].

Definition 1.1.

Let $A \in M_{m,n}(F)$, then the *rank* of the matrix A, denoted by rank(A), is defined as the largest number of columns of A that constitute a linearly independent set. This set of columns is not unique, but the cardinality of this set is unique.

Remark 1.1.

If $A \in M_{m,n}(C)$, then rank(A^*)=rank(A^T)=rank$(\bar{A})$=rank(A).

Lemma 1.1.

Let $A \in M_{m,n}(F)$. *Then rank*$(A) = k$ *if and only if there is a* $k-by-k$ *submatrix of* A *with nonzero determinant, but all* $(k+1)-by-(k+1)$ *submatrices of* A *have determinant zero.*

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Definition 1.2.
Suppose that V is a vector space over F. Then an *inner product* on V is a function $<\cdot,\cdot>: V \times V \longrightarrow F$ such that $\forall\ x,\ y,\ z \in V$ and $\forall\ \alpha, \beta \in F$ satisfies the following:

(i) $< x, x > \geq 0,\ \forall\ x \neq 0$ and $< x, x >= 0$ iff $x = 0$.

(ii) $\overline{< x, y >} =< y, x >$, where the bar denotes the complex conjucate.

(iii) $<\cdot,\cdot>$ is *linear* in the first argument, *i.e.*, $< \alpha x + \beta y, z >= \alpha < x, z > + \beta < y, z >$.

(iv) $<\cdot,\cdot>$ is *conjugate linear* in the second argument, *i.e.*, $< x, \alpha y + \beta z >= \bar{\alpha} < x, y > + \bar{\beta} < x, z >$.

Definition 1.3 (Block diagonal matrices).
A matrix $A \in M_n$ of the form

$$A = \begin{bmatrix} A_{11} & & & \mathbf{0} \\ & A_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & A_{kk} \end{bmatrix}$$

where $A_{ii} \in M_{n_i},\ i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} n_i = n$, is called a *block diagonal matrix*. For such a matrix, it is often indicated as $A = A_{11} \oplus A_{22} \oplus, \ldots, \oplus A_{kk}$, which is known as the *direct sum* of the matrices $A_{11}, \ldots, A_{kk}$. Also $\det(\oplus \sum_{i=1}^{k} A_{ii}) = \prod_{i=1}^{k} \det(A_{ii})$ and $\mathrm{rank}(\oplus \sum_{i=1}^{k} A_{ii}) = \sum_{i=1}^{k} \mathrm{rank}(A_{ii})$.

Definition 1.4 (Block triangular matrices).
A matrix $A \in M_n$ of the form

$$A = \begin{bmatrix} A_{11} & & & * \\ & A_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & A_{kk} \end{bmatrix}$$

where $A_{ii} \in M_{n_i},\ i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} n_i = n$ and "$*$"denotes any entry, is called a *block triangular matrix*.

Definition 1.5 (Permutation matrices).
A matrix $P \in M_n$ is called a *permutation matrix* if exactly one entry in each row and column is equal to 1, and all other entries are 0. For example,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a permutation matrix, and $P \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ is a permutation of the rows of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, namely, the permutation that sends the first item to the second position, second item to first position and leaves the third item in the third position.

Definition 1.6 (Change of basis).
Let V be an n dimensional vector space over the field F. Let $\mathcal{B}_1 = \{e_1, e_2, \ldots, e_n\}$ and $\mathcal{B}_2 = \{f_1, f_2, \ldots, f_n\}$ be two ordered basis of V. Then it is obvious that we can write

$$f_i = \sum_{j=1}^{n} a_{ij} e_j$$

where a_{ij} are called coordinates of f_i w.r.t. $\mathcal{B}_1$. Then we have,

$$P = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

This matrix P is called the *change of basis matrix* from the old basis $\mathcal{B}_1$ to the new basis $\mathcal{B}_2$. Again the columns of P are the coordinate vector of $f_i, i = 1, 2, \dots, n$ w.r.t the basis $\mathcal{B}_1$ *i.e.,* $P_j = [f_j]_{\mathcal{B}_1}, j = 1, \dots, n$.

Theorem 1.1.
Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two ordered basis of a finite dimensional vector space V over F and let $T : V \longrightarrow V$ be linear. Then $\exists$ an invertible matix P such that $[T]_{\mathcal{B}_2} = P^{-1}[T]_{\mathcal{B}_1}P$

The importance of this result is that the matices $[T]_{\mathcal{B}_2}$ and $[T]_{\mathcal{B}_1}$ are similar.

Definition 1.7.
The *spectrum* of $A \in M_n$ is the set of all $\lambda \in C$ that are eigenvalues of A, we denote this set by $\sigma(A)$. And the *spectral radius* of A is the non-negative real number, $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

Definition 1.8.
A matrix $B \in M_n$ is said to be *similar* to a matrix $A \in M_n$, denoted by $B \sim A$, if $\exists$ a nonsingular matrix $S \in M_n$ such that $B = S^{-1}AS$.

Definition 1.9.
A linear operator T on a finite-dimensional vector space V is called *diagonalizable* if $\exists$ an ordered basis $\mathcal{B}$ for V such that $[T]_{\mathcal{B}}$ is diagonal.

Definition 1.10 (Simultaneous Diagonalization).
Two diagonalizable matrices $A, B \in M_n$ are said to be *simultaneously diagonalizable* if there is a single similarity matrix $S \in M_n$ such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal, *i.e.,* if there is a single basis in which the representations are diagonal.

Theorem 1.2.
Let $A, B \in M_n$ be diagonalizable. Then A and B commute iff they are simultaneously diagonalizable.

Definition 1.11.
A *family* $\mathcal{F} \subseteq M_n$ of matrices is an arbitrary(finite or infinite) set of matrices, and a *commuting family* is one in which each pair in the set commutes under multiplication. A subspace $W \subseteq C^n$ is said to be $A-invariant$, for $A \in M_n$, if $Aw \in W$ for every $w \in W$; and W is called $\mathcal{F}-invariant$, for a family $\mathcal{F} \subseteq M_n$, if W is $A-invariant$ for each $A \in \mathcal{F}$.

Lemma 1.2.
If $\mathcal{F} \subseteq M_n$ is a commuting family, then there is a vector $x \in C^n$ which is an eigen vector of every $A \in \mathcal{F}$.

Definition 1.12.
A *simultaneously diagonalizable family* $\mathcal{F} \subseteq M_n$ is a family for which there is a single nonsingular matrix $S \in M_n$ such that $S^{-1}AS$ is diagonal for every $A \in \mathcal{F}$.

Theorem 1.3.
Suppose that $A \in M_{m,n}$ and $B \in M_{n,m}$ with $m \leq n$. Then BA has the same eigenvalues as AB, counting multiplicity, together with an additional $n-m$ eigenvalues equal to 0; i.e., $p_{BA}(t) = t^{n-m}p_{AB}(t)$. If $m = n$ and atleast one of A or B is nonsingular, then AB and BA are similar.

2. Positive Definite Matrices

Definition 2.1.
A matrix $A \in M_n(F)$, where $F = R$ or C, is said to be *positive definite* if $x^*Ax > 0$, $\forall x (\neq 0) \in F^n$ and *positive semi-definite* if $x^*Ax \geq 0$, $\forall x (\neq 0) \in F^n$. Here, $x^* = x^T$, when $F = R$.

The following are very important properties regarding positive definite matrices.

(i) The eigenvalues of a positive definite matrix are positive and that of semi-positive definite matrix are nonnegative.

(ii) If $A \in M_n$ is positive definite, then $\bar{A}, A^T, A^*$ and A^{-1} are also positive definite.

(iii) Any principal submatrix of a positive definite matrix is positive definite.

(iv) The diagonal entries of a positive definite matrix are real numbers.

(v) The sum of any two positive definite matrices of the same size is positive definite.

(vi) The trace, determinant and all principal minors of a positive definite matrix are positive.

Characterizations: Regarding the characterizations of positive definite matrices we have the following small results.

(i) A Hermitian matrix $A \in M_n$ is positive semidefinite iff all of its eigenvalues are nonnegative. It is positive definite iff its eigenvalues are positive.

(ii) If $A \in M_n$ is positive semidefinite, then so are all the powers $A^k, k = 1, \ldots$

(iii) If $A \in M_n$ is Hermitian and strictly diagonally dominant and if $a_{ii} > 0$ for all $i = 1, \ldots, n$, then A is positive definite.

(iv) If $A \in M_n$ is Hermitian, then A is positive definite iff $|A_i| > 0$ for $i = 1, \ldots, n$.

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