Linear Algebra

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Question 1. *Let M_n be the vector space of all $n \times n$ matrices with real entries under usual matrix addition and scalar multiplication. Find the dimensions of the following subspaces of M_n:*

1. $W_1 = \{A \in M_n : A \text{ is symmetric}\}$
2. $W_2 = \{A \in M_n : A \text{ is skew-symmetric}\}$
3. $W_3 = \{A \in M_n : A \text{ is upper triangular}\}$
4. $W_4 = \{A \in M_n : A \text{ is tridiagonal}\}$

Solution. It is to be noted that the dimensions of the above subspaces can be found just by calculating the number of entries which we can choose freely to construct a matrix of the given type. For symmetric matrices, $a_{ij} = a_{ji}$, so that we are not free to choose the $(j,i)th$ entry once we choose the $(i,j)th$ entry i.e. once we choose the $(i,j)th$ entry, the $(j,i)th$ entry is automatically chosen. Thus, the dimension of W_1 is just the number of entries on and above the diagonal of an $n \times n$ matrix. So, $dim(W_1) = n + (n-1) + (n-2) + \ldots + 2 + 1 = \frac{n(n+1)}{2}$.

For skew-symmetric matrices, $a_{ij} = -a_{ji}$ and so the diagonal entries are all zero. Also, as above, the selection of $(i,j)th$ entry finalizes the $(j,i)th$ entry too. So, the dimension of W_2 is just the number of entries above the diagonal of an $n \times n$ matrix. Hence, $dim(W_1) = (n-1) + (n-2) + \ldots + 2 + 1 = \frac{n(n-1)}{2}$.

Similarly, $dim(W_3) = \frac{n(n+1)}{2}$.

Now, for a tridiagonal matrix, the entries in all other positions except in the main diagonal, the first diagonal above it and the first diagonal below it are always zero. So the dimension of W_4 is just the number of entries in these three diagonals of an $n \times n$ tridiagonal matrix. There are two entries each in the first and the last rows and three each in all other rows. So, $dim(W_4) = 2 + 3(n-2) + 2 = 3n - 2$. □

Question 2. *Let V be the subspace of $M_2(\mathbb{R})$ consisting of matrices such that the entries of the first row add up to zero. Write down a basis for V.*

Solution. The matrices in V will be of the form $\begin{pmatrix} a & -a \\ b & c \end{pmatrix}$

Hence, one possible basis is $\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ □

Question 3. *Write down a necessary and sufficient condition, in terms of a, b, c and d (which are assumed to be real numbers), for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ not to have a real eigenvalue.*

Proof. Clearly, this matrix won't have any real eigenvalues if and only if its characteristic polynomial has all imaginary zeros.

Thus, $(a+d)^2 < 4(ad-bc)$. $\square$

Question 4. *Given that the matrix* $\begin{pmatrix} \alpha & 1 \\ 2 & 3 \end{pmatrix}$ *has 1 as an eigenvalue, compute its trace and its determinant.*

Solution. Let the other eigenvalue be x. Then $\alpha + 3 = 1 + x$ and $3\alpha - 2 = x$, so that $\alpha = 2$. so, trace=5 and determinant=4 $\square$

Question 5. *Let A be a non diagonal 2×2 matrix with complex entries such that $A = A^{-1}$. Write down its characteristic and minimal polynomials.*

Solution. Since $A = A^{-1}$, so $det(A) = \pm 1$. But, for A to be non-diagonal matrix, $det(A) = -1$. Thus, A is of the form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ where $ab = 1$. Since $A^2 = I$, so the characteristic polynomial of A is $x^2 - 1$ and the minimal polynomial is also $x^2 - 1$. $\square$

Question 6. *For a fixed positive integer $n \geq 3$, let A be the $n \times n$ matrix defined by $A = I - \frac{1}{n}J$, where J is the $n \times n$ matrix with all entries equal to 1. Which of the following statements is not true?*

1. *$A^k = A$ for every positive integer k.*
2. *$Trace(A) = n - 1$*
3. *$Rank(A) + Rank(I - A) = n$*
4. *A is invertible.*

Solution. Simple matrix multiplication reveals that $A^2 = A$ and hence it can be shown that $A^k = A$ for every positive integer k, so that (1) is true.

The simplest thing that we can check is

$$Trace(A) = Trace(I) - Trace\left(\frac{1}{n}J\right) = n - 1$$

Hence (2) is true.

Now, since $A = I - \frac{1}{n}J$, so $I - A = \frac{1}{n}J$ and so $rank(I - A) = 1$. Also, performing the elementary operations $R_n \to R_n + R_n - 1 + \ldots + R_1$ and then $R_i \to R_i - R_1 \forall i = 2, 3, \ldots, n-1$, it is clear that $Rank(A) = n - 1$ and so (3) is true.

Now, it can be seen that $x = (1, 1, \ldots, 1)^T_{1 \times n}$ satisfies $Ax = 0$, so that 0 is an eigenvalue of A. Hence A is singular and so A is not invertible. So, (4) is not true. $\square$

Question 7. *Let A be a 5×4 matrix with real entries such that $Ax = 0$ if and only if $x = 0$, where x is a 4×1 vector. Then find the rank of A.*

Solution. From the question it is clear that the homogeneous system of equations $Ax = 0$ has only trivial solution. So, $Rank(A) =$ the number of unknowns $= 4$. $\square$

Question 8. *Consider the following row vectors*

$$a_1 = (1,1,0,1,0,0), a_2 = (1,1,0,0,1,0), a_3 = (1,1,0,0,0,1)$$

$$a_4 = (1,0,1,1,0,0), a_5 = (1,0,1,0,1,0), a_6 = (1,0,1,0,0,1)$$

What is the dimension of the real vector space spanned by these vectors?

Solution. Simple elementary operations will reveal that only four of these vectors are linearly independent. So, dimension of the vector space spanned by these vectors is 4. □

Question 9. *Let* $A = (a_{ij})_{n\times n}, n \geq 3$ *where* $a_{ij} = b_i^2 - b_j^2, i, j = 1, 2, \ldots, n$ *for some distinct real numbers* $b_1, b_2, \ldots, b_n$. *Then find* $det(A)$.

Solution. Performing the elementary operations, $R_i \to R_i - R_1$ for each $i = 2, 3, \ldots, n$ we see that $det(A) = 0$. Note that $det(A)$ may not be zero if $n = 2$. □

Question 10. *Number of matrices with real entries, of the form* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *such that* $A^2 - A = 0$ *is*

1. *only one*
2. *only two*
3. *finitely many*
4. *infinitely many*

Solution. Using the equation $A^2 = A$, it can be shown that any matrix A of the form $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ where $bc = a - a^2$, satisfies the given condition. So, there are infinitely many such matrices. □

Question 11. *Let* V *be a vector space of dimension* n *over the field* $\mathbb{Z}_p$ *where* p *is a prime. Then how many elements are there in* V*?*

Solution. Since $dim(V) = n$, so let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V. Then any element of V is a linear combination of the form $\sum_{i=1}^{n} \alpha_i e_i$ where $\alpha_i \in \mathbb{Z}_p$. So, the number of elements in V is p^n. □

Question 12. *If* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *be a linear transformation defined by* $T(1,0,0) = (1,0,1)$*,* $T(0,1,0) = (0,0,1)$ *and* $T(0,0,1) = (1,0,0)$*. Then find the range space of* T*, null space of* T*, rank and nullity of* T*.*

Solution. Clearly, $T(x,y,z) = xT(1,0,0)+yT(0,1,0)+zT(0,0,1) = (x+z, 0, x+y) = (x+z)(1,0,0) + (x+y)(0,0,1)$. Thus the range space of T is $\{(x,0,y) : x, y \in \mathbb{R}\}$. Also, $T(x,y,z) = 0 \Rightarrow x = -y = -z$ and so the null space of T is $\{(x,-x,-x) : x \in \mathbb{R}\}$. Rank($T$)= Dimension of the range space=2 and nullity(T)=Dimension of the null space of T=1. □

Question 13. *Let* $A \in M_2(\mathbb{R})$ *such that* $tr(A) = 2$ *and* $det(A) = 3$*. Write down the characteristic polynomial of* A^{-1}*.*

Solution. Here, the characteristic polynomial of A is x^2-(sum of eigenvalues)$x+$ (product of eigenvalues) i.e $x^2-tr(A)x+det(A)$. By Cayley Hamilton Theorem,

$$\begin{aligned}&A^2 - tr(A)A + det(A)I = 0\\ \Rightarrow &A^2 - 2A + 3I = 0\\ \Rightarrow &(A^{-1})^2 - \frac{2}{3}A^{-1} + \frac{1}{3}I = 0\end{aligned}$$

Thus the characteristic polynomial of A^{-1} is $x^2 - \frac{2}{3}x + \frac{1}{3}$. □

Question 14. *Let a 3×3 matrix A have eigenvalues $1, 2, -1$, then find the trace of the matrix $B = A - A^{-1} + A^2$.*

Solution. By Cayley-Hamilton theorem, $(A^2 - 1)(A - 2) = 0 \Rightarrow A^3 - 2A^2 - A + 2I = 0 \Rightarrow A^2 - 2A - I + 2A^{-1} = 0 \Rightarrow A - A^{-1} = \frac{1}{2}(A^2 - I)$. So, $B = A - A^{-1} + A^2 = \frac{3}{2}A^2 - \frac{1}{2}I$. Thus, eigenvalues of B are $\frac{3}{2} \times 1^2 - \frac{1}{2} = 1, \frac{3}{2}\times 2^2 - \frac{1}{2} = \frac{11}{2}, \frac{3}{2}\times(-1)^2 - \frac{1}{2} = 1$, so that the trace$(B) = 1 + \frac{11}{2} + 1 = \frac{15}{2}$ □

Question 15. *The eigenvectors of a 3×3 matrix A corresponding to the eigenvalues $1, 1, 2$ are $(1, 0, -1)^T, (0, 1, -1)^T$ and $(1, 1, 0)^T$. Find the matrix A.*

Solution. Let $A = \begin{pmatrix} a & b & c\\ u & v & w\\ x & y & z\end{pmatrix}$. Then, we can compute the unknowns using the relations $A\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$, $A\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$ and $A\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$. It can be shown that $A = \begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 0 & 0 & 1\end{pmatrix}$ □

Question 16. *What is the null space of the matrix $A = \begin{pmatrix}1 & 0\\ 2 & 5\\ 1 & 4\end{pmatrix}$?*

Solution. Its easy to see that the rank of A is 2 and so by rank-nullity theorem, the nullity of A is 0, so that the null space of A is the zero space. □

Question 17. *Let V be the vector of all $n \times n$ matrices over the field $\mathbb{R}$ of real numbers. A subset W of V is not a subspace of V if W consists of all matrices for which*

1. $A^2 = A$
2. $A^2 \neq A$
3. $A = A^T$
4. $A = -A^T$

Solution. Clearly, under the restriction $A^2 \neq A$, the zero vector cannot belong to W and so W cannot be a subspace of V. $\square$

Question 18. *Let A be a 4×4 invertible matrix. Which of the following is NOT true?*

1. *The rows of A form a basis of $\mathbb{R}^4$*
2. *Null space of A contains only the zero vector*
3. *A has four distinct eigenvalues*
4. *Image of the linear transformation $x \mapsto Ax$ on $\mathbb{R}^4$ is $\mathbb{R}^4$.*

Solution. Since A is invertible, so $rank(A) = 4$ and so the rows of A are linearly independent and hence they form a basis of $\mathbb{R}^4$. Since $nullity(A) = 0$, so the null space of A contains only the zero vector. It can be seen that for $e_1 = (1, 0, 0, 0)^T, e_2 = (0, 1, 0, 0)^T, e_3 = (0, 0, 1, 0)^T, e_4 = (0, 0, 0, 1)^T$, their images Ae_1, Ae_2, Ae_3 and Ae_4 are the columns of A, which are linearly independent as $rank(A) = 4$. But it is not necessary that A has distinct eigenvalues. Thus option 3 is NOT correct. $\square$

Question 19. *Let $C(\mathbb{R})$ be the linear space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let W_c be the set of differentiable functions $u(x)$ that satisfy the differential equation $u' = xu + c$. For which value(s) of the real constant c is this set a linear subspace of $C(\mathbb{R})$?*

Solution. Clearly if W_c is a linear subspace then $u+v \in W_c$ whenever $u, v \in W_c$, so that $u + v$ should also satisfy the given differential equation. So, $u' = xu + c, v' = xv + c$ and $u' + v' = x(u + v) + c$. Hence, $2c = c \Rightarrow c = 0$. $\square$

Question 20. *Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$ and $U \neq V$, and let $W = U \cap V$. Find all possible values for the dimension of W.*

Solution. Here, $dim(U) = 2$ and $dim(V) = 2$. Let $B_U = \{e_1, e_2\}$ and $B_V = \{e_3, e_4\}$ be bases of U and V respectively. It may happen that $\{e_1, e_2, e_3, e_4\}$ is LI, in which case $dim(U + V) = 4 \Rightarrow dim(U) + dim(V) - dim(U \cap V) = 4 \Rightarrow dim(U \cap V) = 0$.

Again, it may happen that $\{e_1, e_2, e_3, e_4\}$ is LD but $\{e_1, e_2, e_3\}$ is LI, in which case $dim(U + V) = 3 \Rightarrow dim(U \cap V) = 1$. $\square$

Question 21. *Let P_n be the vector space of all polynomials of degree at most $n \geq 6$ with real coefficients. Consider the following subspaces of P_n:*

$$W_1 = \{p(x) \in P_n : p(1) = 0, p(2) = 0, p(3) = 0, p(4) = 0\}$$

$$W_2 = \{p(x) \in P_n : p(3) = 0, p(4) = 0, p(5) = 0, p(6) = 0\}$$

Find the dimensions of W_1, W_2 and $W_1 \cap W_2$

Solution. Here,

$$W_1 = \{p(x) \in P_n : p(x) = (x - 1)(x - 2)(x - 3)(x - 4)q(x) \text{ where } q(x) \in P_{n-4}\}$$

$$W_1 = \{p(x) \in P_n : p(x) = (x - 3)(x - 4)(x - 5)(x - 6)q(x) \text{ where } q(x) \in P_{n-4}\}$$

Thus, $dim(W_1) = dim(W_2) = n-3$. Now, $W_1 \cap W_2$ contains all polynomials of the form

$$p(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)q(x), \text{ where } q(x) \in P_{n-6}$$

Thus, $dim(W_1 \cap W_2) = n-5$ ☐

Question 22. *Let $\mathcal{P}_3$ denote the real vector space of all polynomials with real coefficients having degree less than or equal to 3, equipped with the standard ordered basis $\{1, x, x^2, x^3\}$. Write down the matrix, w.r.t this basis,of the following linear transformation:*

$$L(p) = p'' - 2p' + p, p \in \mathcal{P}_3$$

Also, find the unique polynomial p such that $L(p) = x^3$.

Solution. Here,

$$\begin{aligned} L(1) &= 1 && = 1.1 + 0.x + 0.x^2 + 0.x^3 \\ L(x) &= x - 2 && = (-2).1 + 1.x + 0.x^2 + 0.x^3 \\ L(x^2) &= x^2 - 4x + 2 && = (2).1 + (-4).x + 1.x^2 + 0.x^3 \\ L(x^3) &= x^3 - 6x^2 + 6x && = 0.1 + 6.x + (-6).x^2 + 1.x^3 \end{aligned}$$

Thus matrix of $L = \begin{pmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -4 & -6 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Now let $p = ax^3 + bx^2 + cx + d \in \mathcal{P}_3$ be such the $L(p) = x^3$. Then, using the definition of L and equating the coefficients on both sides, we get $p = x^3 + 6x^2 + 18x + 24$. ☐

Question 23. *A non-zero matrix $A \in M_n(\mathbb{R})$ is said to be nilpotent if $A^k = 0$ for some positive integer $k \geq 2$. If A is nilpotent, which of the following statements are true?*

1. *$k \leq n$ for the smallest such k.*
2. *The matrix $I + A$ is invertible.*
3. *All the eigenvalues of A are zero.*

Solution. Clearly, A^k has all its eigenvalues equal to zero. But eigenvalues of A^k are just the kth power of the eigenvalues of A. Thus, all the eigenvalues of A must be zero. Hence (3) is true. The eigenvalues of $I+A$ are $1+0, 1+0, 1+0$ i.e. $1, 1, 1$ and so it is invertible. In fact $(I+A)^{-1} = I - A + A^2 - A^3 + \ldots + (-1)^{k-1}A^k$. Hence, (2) is true. Since all the eigenvalues of A are zero, so the characteristic equation of A is $x^n = 0$ and so by Cayley Hamilton theorem, $A^n = 0$. Thus, the smallest such k for which $A^k = 0$ can at most be equal to n. Thus, $k \leq n$. Hence, (1) is also true. ☐

Question 24. *Justify whether the following are true or false:*

1. *There exist $n \times n$ matrices A and B such that $AB - BA = I$.*

2. *Let A and B be two arbitrary $n \times n$ matrices. If B is invertible, then $tr(A) = tr^{-1}(B - 1AB)$ where $tr(M)$ denotes the trace of an $n \times n$ matrix M.*

Solution. (1) is false. Suppose there exist such matrices. Then $tr(AB - BA) = tr(I) \Rightarrow tr(AB) - tr(BA) = n$. But $tr(AB) = tr(BA)$ and so we would end up with $0 = n$ which is absurd. Hence there cannot exist such matrices.

(2) is true. This is because similar matrices have same eigenvalues and here A and $B^{-1}AB$ are similar. Thus, $tr(A) =$ sum of the eigenvalues $= tr(B^{-1}AB)$ ☐

Question 25. *Let V be the real vector space of all polynomials in one variable with real coefficients and of degree less than, or equal to 5. Let W be the subspace defined by $W = \{p \in V | p(1) = p'(2) = 0\}$. What is the dimension of W?*

Solution. Since $p(1) = 0$, so $(x - 1)$ is a factor of $p(x) \in V$. thus, $p(x) = (x - 1)q(x)$ where $deg(q(x)) \leq 4$. Now, $p'(2) = 0$ implies that $q(2) + q'(2) = 0$. Now, $q(x) = (x - 2)g(x) + q(2)$ where $deg(g(x)) \leq 3$. Thus, the elements in W are of the form $p(x) = (x - 1)(x - 2)g(x) + q(2)(x - 1)$. But since W is a subspace, so we must have $q(2) = 0$. Thus, $q'(2) = 0$ and any element of W is of the form $p(x) = (x - 1)(x - 2)g(x)$, where $g(x)$ is a polynomial of degree at most 3. So, dimension of W is 4. ☐