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A Brief Introduction to Tensor

Introductory Article

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Abstract: In this article, our aim is to give a brief idea about the *tensors*. We will discuss about their different types alongwith some examples, their properties, relevant proofs, etc. We will also discuss some special tensors like *Kronecker delta, Levi-Civita symbol, null tensor, metric tensor, inertia tensor, susceptibility* χ etc.

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1. Introduction and Motivation

What is a tensor? How the concept of tensor comes to be a part of mathematics? Why we need to study the tensor algebra? What is the physical motivation? Well, being frequent to all, although this should not be an ideal starter, but situation demands the answers. The first question could hardly be more straightforward, and yet I have been woefully unsuccessful at providing anything resembling a satisfactory answer. Even a partial answer that elicited a follow-up question would rate as a success, but this is rare, especially so in mixed company at cocktail parties. Why is it that the simplest questions are so often the hardest to answer? This note explains my current thoughts on the topic, and may serve as a guide should someone be foolish enough to raise the question at a future occasion.

As every successful politician knows, the essential first step in answering any difficult question is to 're-phrase the question'. This can mean anything from 'the question I think was intended' to 'the question

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I know how to answer' to 'the question I want to be asked'. Even a straightforward question, such as 'what is a tensor?' is suspectible to this ploy. Here are some re-phrased versions of the question, roughly increasing order of difficulty:

1. What is the mathematical definition of a tensor?
2. Is a tensor a kind of a vector?
3. Is a vector a kind of a tensor?
4. Is a matrix a special kind of a tensor?
5. When you write λ^i, do you mean a row vector or a column vector?
6. What are tensors used for?
7. I know that a vector has magnitude and direction, but what does a tensor look like?

The first of these questions is easily disposed of by the stock answer, courtesy of vector spaces, 'a tensor is an element of a tensor space'. The second and third questions are answered in a single word, 'yes'. Question 4 is only slightly more difficult, the answer being 'yes and no'. Question 5 deals with typographical strategy: it all depends on whether the elements are written across the page, down the page, or diagonally.

Audiences are invariably nonplussed by the directness and simplicity of these answers. The obvious correctness of the answer to the question 1 is eloquent testimony to the persuasive power of modern mathematics. What more is there to say?

Regarding question 7, one could preface a reply by stating that vectors need have neither magnitude nor direction, and the same is true for tensors. But that, I suspect, is more likely to confuse than to enlighten.

Well, now come to the second and third question of the begining, *i.e., How the concept of tensor comes to be a part of mathematics? Why we need to study the tensor algebra?* A simple answer to these questions seems like as follows.

Tensors are a generalisation of vectors. We think informally of a tensor as something which, like a vector, can be measured component-wise in any Cartesian frame; and which also has a physical significance independent of the frame, like a vector.

Regarding the physical motivation, we will try to give a suitable answer later on in the next section. More information regarding tensors can be obtained in [1] or [2].

2. Technical details

Here, in this section, we will try to give the formal definition of a tensor, alongwith its types, rank, properties etc. But, as we have already mention that tensors are a generalisation of vectors, so before proceeding towards the main goal, we try to give a very brief idea about the vectors.

2.1. Vectors

A vector is more than just 3 real numbers. It is also a physical entity: if we know its 3 components with respect to one set of Cartesian axes then we know its components with respect to any other set of Cartesian axes. (The *vector* stays the same even if its components do not).

For example, suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a right-handed orthogonal set of unit vectors, and that a vector $\mathbf{v}$ has components vi relative to axes along those vectors. That is to say,

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_j\mathbf{e}_j.$$

What are the components of $\mathbf{v}$ with respect to axes which have been rotated to align with a different set of unit vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$? Let

$$\mathbf{v} = v'_1\mathbf{e}'_1 + v'_2\mathbf{e}'_2 + v'_3\mathbf{e}'_3 = v'_j\mathbf{e}'_j.$$

Now, $\mathbf{e}'_i.\mathbf{e}'_j = \delta_{ij}$, so

$$\mathbf{v}.\mathbf{e}'_i = v'_j\mathbf{e}'_j.\mathbf{e}'_i = v'_j\delta_{ij} = v'_i$$

but also

$$\mathbf{v}.\mathbf{e}'_i = v_j\mathbf{e}_j.\mathbf{e}'_i = v_jl_{ij}$$

where we define the matrix $L = (l_{ij})$ by

$$l_{ij} = \mathbf{e}'_i.\mathbf{e}_j.$$

Then

$$v'_i = l_{ij}v_j.$$

(or, in matrix notation, $\mathbf{v}' = L\mathbf{v}$ where $\mathbf{v}'$ is the column vector with components v'_i). L is called the *rotation matrix* or *transformation matrix*.

Note 2.1.
This *looks* like, but is *not quite the same as*, rotating the vector $\mathbf{v}$ round to a different vector $\mathbf{v}$ using a transformation matrix L. In the present case, $\mathbf{v}$ and $\mathbf{v}'$ are the *same* vector, just measured with respect to different axes. The transformation matrix corresponding to the rotation $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \mapsto \{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ is not L (in fact it is L^{-1}).

Now, we are in a position to enter into our main objective, *i.e.,* Tensor.

2.2. Tensors

Tensors are nothing but the mathematical objects, which transforms like a coordinate transformation. Now, its time to elaborate what is the physical motivation of a tensor.

2.2.1. Physical Motivation

We know that the conductivity law, $\mathbf{J} = \sigma\mathbf{E}$, where $\mathbf{E}$ is the applied electric field and $\mathbf{J}$ is the resulting electric current. This is suitable for simple isotropic media, where the conductivity is the same in all directions. But a matrix formulation may be more suitable in anisotropic media; for example,

$$\mathbf{J} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{E}$$

might represent a medium in which the conductivity is high in the x-direction but in which no current at all can flow in the z-direction. (For instance, a crystalline lattice structure where vertical layers are electrically insulated). More generally, in suffix notation we have $J_i = \sigma_{ij}E_j$, where σ is the *conductivity tensor*.

What happens if we measure $\mathbf{J}$ and $\mathbf{E}$ with respect to a different set of axes? We would expect the matrix σ to change too: let its new components be σ_{ij} . Then $J_i' = \sigma_{ij}'E_j'$. But $\mathbf{J}$ and $\mathbf{E}$ are vectors, so $J_i' = l_{ij}J_j$ and $E_i = l_{ji}E_j'$ from the results regarding the transformation of vectors in §2.1. Hence,

$$\begin{aligned}
\sigma_{ij}'E_j' &= J_i' \\
&= l_{ip}J_p \\
&= l_{ip}\sigma_{pq}E_q \\
&= l_{ip}\sigma_{pq}l_{jq}E_j' \\
\Rightarrow (\sigma_{ij}' - l_{ip}l_{jq}\sigma_{pq})E_j' &= 0.
\end{aligned}$$

This is true $\forall$ vectors $\mathbf{E}'$, and hence the bracket must be identically zero; hence $\sigma'_{ij} = l_{ip}l_{jq}\sigma_{pq}$. This tells us how σ transforms. Here σ is a second rank tensor, because it has two suffixes (σ_{ij}).

2.2.2. Definition of Tensor

Definition 2.1.
In general, a tensor of *rank* n is a mathematical object with n suffixes, $T_{i_1 i_2 \ldots i_n}$, which obeys the transformation law

$$T'_{i_1 i_2 \ldots i_n} = l_{i_1 p_1} l_{i_2 p_2} \ldots l_{i_n p_n} T_{i_1 i_2 \ldots i_n}$$

where L is the rotation matrix between frames.

The above definition can be given alternatively as follows.

Definition 2.2.
A tensor of *rank* n is a mathematical object with n suffixes, $T_{\alpha_1 \alpha_2 \ldots \alpha_n}$, which follows the transformation law

$$T'_{\alpha_1 \alpha_2 \ldots \alpha_n} = \frac{\partial x^{\mu_1}}{\partial x'^{\alpha_1}} \frac{\partial x^{\mu_2}}{\partial x'^{\alpha_2}} \cdots \frac{\partial x^{\mu_n}}{\partial x'^{\alpha_n}} T_{\mu_1 \mu_2 \ldots \mu_n}$$

where $\frac{\partial x^{\mu_1}}{\partial x'^{\alpha_1}} \frac{\partial x^{\mu_2}}{\partial x'^{\alpha_2}} \cdots \frac{\partial x^{\mu_n}}{\partial x'^{\alpha_n}}$ is called the transformation machinary.

For example, a second rank tensor with free indices α, β is given by the following transformation law

$$T'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} T_{\mu\nu}.$$

2.2.3. Types of Tensor

There are three types of tensors, viz. *Contravariant Tensors, Covariant Tensors* and *Mixed Tensors*.

(i) *Contravariant Tensors*: A contravariant tensor of rank n with free indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ follows the following transformation law

$$T'^{\alpha_1 \alpha_2 \ldots \alpha_n} = \frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}} \frac{\partial x'^{\alpha_2}}{\partial x^{\mu_2}} \cdots \frac{\partial x'^{\alpha_n}}{\partial x^{\mu_n}} T^{\mu_1 \mu_2 \ldots \mu_n}$$

where $\frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}} \frac{\partial x'^{\alpha_2}}{\partial x^{\mu_2}} \cdots \frac{\partial x'^{\alpha_n}}{\partial x^{\mu_n}}$ is the transformation machinary.

(ii) *Covariant Tensors*: A covariant tensor of rank n with free indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ follows the following transformation law

$$T'_{\alpha_1 \alpha_2 \ldots \alpha_n} = \frac{\partial x^{\mu_1}}{\partial x'^{\alpha_1}} \frac{\partial x^{\mu_2}}{\partial x'^{\alpha_2}} \cdots \frac{\partial x^{\mu_n}}{\partial x'^{\alpha_n}} T_{\mu_1 \mu_2 \ldots \mu_n}$$

where $\frac{\partial x^{\mu_1}}{\partial x'^{\alpha_1}} \frac{\partial x^{\mu_2}}{\partial x'^{\alpha_2}} \cdots \frac{\partial x^{\mu_n}}{\partial x'^{\alpha_n}}$ is the transformation machinary.

(iii) *Mixed Tensors*: A mixed tensor of rank n with contravariant free indices $\alpha_1, \alpha_2, \ldots, \alpha_i$ and covariant free indices $\beta_1, \beta_2, \ldots, \beta_j$, with $i + j = n$ follows the following transformation law

$$T'^{\alpha_1\alpha_2\ldots\alpha_i}_{\beta_1\beta_2\ldots\beta_j} = \frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}}\frac{\partial x'^{\alpha_2}}{\partial x^{\mu_2}}\cdots\frac{\partial x'^{\alpha_i}}{\partial x^{\mu_i}}\frac{\partial x^{\nu_1}}{\partial x'^{\beta_1}}\frac{\partial x^{\nu_2}}{\partial x'^{\beta_2}}\cdots\frac{\partial x^{\nu_j}}{\partial x'^{\beta_j}}T^{\mu_1\mu_2\ldots\mu_i}_{\nu_1\nu_2\ldots\nu_j}$$

where $\frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}}\frac{\partial x'^{\alpha_2}}{\partial x^{\mu_2}}\cdots\frac{\partial x'^{\alpha_i}}{\partial x^{\mu_i}}\frac{\partial x^{\nu_1}}{\partial x'^{\beta_1}}\frac{\partial x^{\nu_2}}{\partial x'^{\beta_2}}\cdots\frac{\partial x^{\nu_j}}{\partial x'^{\beta_j}}$ is the transformation machinary.

2.2.4. Examples of Tensor

Here we put some example of tensors in order to illustrate the above definition.

Example 2.1.
Any scalar is a tensor of rank 0. For example, temperature T is a scalar quantity, because it is the same in all frames ($T' = T$). So it is a tensor of rank 0.

Example 2.2.
Any vector is a tensor of rank 1, follows the following transformation law,

$$T'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}}T^{\beta}.$$

Example 2.3.
Any matrix is a tensor of rank 2, follows the following transformation law,

$$T'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}T^{\mu\nu}.$$

2.3. Special Tensors

2.3.1. The Inertia Tensor

Consider a mass m which is part of a rigid body, at a location $\mathbf{x}$ within the body. If the body is rotating with angular velocity ω then the masss velocity is $\mathbf{v} = \omega \times \mathbf{x}$, and its angular momentum is therefore

$$m\mathbf{x} \times \mathbf{v} = m\mathbf{x} \times (\omega \times \mathbf{x}) = m(|\mathbf{x}|^2\omega - (\omega.\mathbf{x})\mathbf{x}).$$

Changing from a single mass m to a continuous mass distribution with density $\rho(\mathbf{x})$, so that an infinitesimal mass element is $\rho(\mathbf{x})dV$, we see that the total angular momentum of a rigid body V is given by

$$\mathbf{h} = \iiint_V \rho(\mathbf{x})(|\mathbf{x}|^2\omega - (\omega.\mathbf{x})\mathbf{x})\mathbf{dV},$$

or, in suffix notation,

$$\begin{aligned} h_i &= \iiint_V \rho(\mathbf{x})(x_k x_k \omega_i - \omega_j x_j x_i) dV \\ &= \iiint_V \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_j x_i)\omega_j dV \\ &= I_{ij}\omega_j, \end{aligned}$$

where

$$I_{ij} = \iiint_V \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j) dV$$

is the *inertia tensor* of the rigid body. Note that the tensor I does not depend on ω, only on properties of the body itself; so it may be calculated once and for all for any given body. To see that it is indeed a tensor, note that both $\mathbf{h}$ and ω are vectors, and apply arguments previously used for the conductivity tensor.

2.3.2. Susceptibility χ

If $\mathbf{M}$ is the magnetization (magnetic moment per unit volume) and $\mathbf{B}$ is the applied magnetic field, then for a simple medium, we have $\mathbf{M} = \chi^{(m)}\mathbf{B}$, where $\chi^{(m)}$ is the magnetic susceptibility. This generalises to $M_i = \chi_{ij}^{(m)} B_j$, where $\chi_{ij}^{(m)}$ is the *magnetic susceptibility tensor*. Similarly for polarization density $\mathbf{P}$ in a dielectric: $P_i = \chi_{ij}^{(e)} E_j$, where $\mathbf{E}$ is the electric field and $\chi_{ij}^{(e)}$ is the *electric susceptibility tensor*.

2.3.3. The Kronecker Delta

The *Kronecker delta* is defined as,

$$\delta^i_j = \begin{cases} 1, & \text{if } \ i = j \\ 0, & \text{if } \ i \neq j \end{cases}$$

In n dimensional space, we have $\delta^i_i = \delta^1_1 + \delta^2_2 + \cdots + \delta^n_n = n$. Again, we have

$$\frac{\partial x^i}{\partial x^j} = \begin{cases} 1, & \text{if } \ i = j \\ 0, & \text{if } \ i \neq j \end{cases}$$

Thus, we see that the *Kronecker delta* also can be defined as

$$\delta^i_j = \frac{\partial x^i}{\partial x^j}$$

Similarly, we have

$$\frac{\partial x'^i}{\partial x'^j} = \begin{cases} 1, & \text{if } \; i = j \\ 0, & \text{if } \; i \neq j \end{cases}$$

and so

$$\delta^i_j = \frac{\partial x'^i}{\partial x'^j}$$

From above, it is clear that the *Kronecker delta* is defined in same coordinate system and it is same in all frames ($\delta'^i_j \equiv \delta^i_j$).

Remark 2.1.
When we are defining in different coordinate system, then it is said to be *transformation machinary*, e.g.,

$$T'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}T^{\mu\nu}.$$

2.3.4. Levi-Civita Symbol

Definition 2.3.
In 3 dimensional space, the *Levi-Civita symbol* is defined as follows,

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if} \quad i, j, k \text{ are cyclic} \\ -1, & \text{if} \quad i, j, k \text{ are anticyclic} \\ 0, & \text{if} \quad \text{any of } i, j, k \text{ coincides.} \end{cases}$$

For example, $\epsilon_{123} = 1,\ \epsilon_{132} = -1,\ \epsilon_{312} = 1,\ \epsilon_{213} = -1,\ \epsilon_{112} = 0,\ \epsilon_{121} = 0,\ \epsilon_{133} = 0$. We have

$$\begin{aligned} \epsilon_{1jk}A^jB^k &= \epsilon_{1j1}A^jB^1 + \epsilon_{1j2}A^jB^2 + \epsilon_{1j3}A^jB^3 \\ &= \epsilon_{111}A^1B^1 + \epsilon_{121}A^2B^1 + \epsilon_{131}A^3B^1 + \epsilon_{112}A^1B^2 + \epsilon_{122}A^2B^2 \\ &+ \epsilon_{132}A^3B^2 + \epsilon_{113}A^1B^3 + \epsilon_{123}A^2B^3 + \epsilon_{133}A^3B^3 \\ &= A^2B^3 - A^3B^2 \end{aligned}$$

which is nothing but the 1st component of curl of $\vec{A}$ and $\vec{B}$. Similarly, we have $\epsilon_{2jk}A^jB^k = A^3B^1 - A^1B^3$, which is the 2nd component of curl of $\vec{A}$ and $\vec{B}$ and $\epsilon_{3jk}A^jB^k = A^1B^2 - A^2B^1$, which is the 3rd component of curl of $\vec{A}$ and $\vec{B}$.

Remark 2.2.
Thus we see that in 3 dimensional space, $\epsilon_{ijk}A^jB^k$ gives us the i^{th} component of curl of $\vec{A}$ and $\vec{B}$.

Definition 2.4.
The *Levi-Civita symbol*, in 4 dimensional space, is defined as follows,

$$\epsilon_{ijkl} = \begin{cases} 1, & \text{if } (i,j,k,l) \text{ is an even permutation of } (1,2,3,4) \\ -1, & \text{if } (i,j,k,l) \text{ is an odd permutation of } (1,2,3,4) \\ 0, & \text{if any of } i,j,k \text{ or } l \text{ coincides.} \end{cases}$$

Definition 2.5.
In n dimension, the definition looks like as follows,

$$\epsilon_{a_1 a_2 \ldots a_n} = \begin{cases} 1, & \text{if } (a_1, a_2, \ldots, a_n) \text{ is an even permutation of } (1,2,3,\ldots,n) \\ -1, & \text{if } a_1, a_2, \ldots, a_n \text{ is an odd permutation of } (1,2,3,\ldots,n) \\ 0, & \text{if any of } a_1, a_2, \ldots, a_n \text{ coincides.} \end{cases}$$

2.3.5. The Null Tensor

A *null tensor* is a tensor where all the entries are zero. For example, *null tensor* of rank 1 are those, where all the entries $A^1, A^2, A^3, \ldots$ are zero.

2.3.6. The Stress and Strain Tensors

In an elastic body, stresses (forces) produce displacements of small volume elements within the body. Let this displacement at a location $\mathbf{x}$ be $\mathbf{u}$; then the *strain tensor* is defined to be

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The stress tensor p_{ij} is defined as the j^{th} component of the forces within the body acting on an imaginary plane perpendicular to the i^{th} axis. Hookes law for simple media says that stress $\propto$ strain. We can now generalise this to the tensor formulation

$$p_{ij} = k_{ijkl} e_{kl}$$

where k_{ijkl} is a fourth rank tensor, which expresses the linear (but possibly anisotropic) relationship between p and e.

2.3.7. The Metric Tensor

We know that the *line element* can be defined as follows

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

where $g_{\mu\nu}$ is called the *metric tensor*. Clearly $g_{\mu\nu}$ is symmetric. Thus, we have the following definition.

Definition 2.6.
A *metric tensor* is a second rank covariant symmetric tensor which descrives the nature of the curved space. This space is known as *Riemannian space.* In other sense, *metric tensor* is a tensor which is defined in *Riemannian space.*

2.4. Properties of Tensors

2.4.1. Linear Combination of Tensors

If A_{ij} and B_{ij} are second rank tensors, and $\alpha,\ \beta$ are scalars, then $C_{ij} = \alpha A_{ij} + \beta B_{ij}$ is also a tensor of second rank.

Proof. We know that,

$$
\begin{aligned}
C'_{ij} &= \alpha' A'_{ij} + \beta' B'_{ij} \\
&= \alpha \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} A_{mn} + \beta \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} B_{mn} \\
&= \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} (\alpha A_{mn} + \beta B_{mn}) \\
&= \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} C_{mn}
\end{aligned}
$$

as required.

This result clearly extends to tensors of rank n. □

2.4.2. Symmetric and Anti-Symmetric Tensors

A tensor $T_{ijk\ldots}$ is said to be *symmetric* in a pair of indices (say i, j) if

$$T_{ijk\ldots} = T_{jik\ldots}$$

and *anti-symmetric* in i, j if

$$T_{ijk\ldots} = -T_{jik\ldots}$$

For a second rank tensor we need not specify the indices as there are only two to choose from! For example, δ_{ij} is symmetric; ϵ_{ijk} is anti-symmetric in any pair of indices.

Note 2.2.
Symmetricity is defined explicitely with respect to either *covariant* indices or *contravariant* indices. Symmetricity is not defined with respect to *mixed* indices. If a tensor is symmetric with respect to all indices, then the tensor is said to be *tensor* or *fully tensor*, otherwise we say symmetric with respect to some indices.

Remark 2.3.
If A_{ij} is a symmetric second rank tensor then the matrix corresponding to A is symmetric, *i.e.,* $A = A^T$. Similarly for an anti-symmetric tensor.

Some important properties of tensors regarding symmetricity and antisymmetricity are the following:

(i) Suppose that S_{ij} is a symmetric tensor and A_{ij} an anti-symmetric tensor. Then $S_{ij}A_{ij} = 0$.

Proof. We have

$$S_{ij}A_{ij} = -S_{ij}A_{ji} = -S_{ji}A_{ji} = -S_{ij}A_{ij} \text{ (swapping dummy } i \text{ and } j)$$

$$\Rightarrow 2S_{ij}A_{ij} = 0,$$

as required. □

(ii) Symmetricity is preserved under general coordinate transformation.

Proof. Let A^{ij} be a second rank contravariant tensor, and symmetric, *i.e,*

$$A'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^i}\frac{\partial x'^{\beta}}{\partial x^j}A^{ij}$$

and $A^{ij} = A^{ji}$. Now

$$A'^{\beta\alpha} = \frac{\partial x'^{\beta}}{\partial x^m}\frac{\partial x'^{\alpha}}{\partial x^n}A^{mn}$$

Let us change m to j and n to i, then we have

$$\begin{aligned} A'^{\beta\alpha} &= \frac{\partial x'^{\beta}}{\partial x^j}\frac{\partial x'^{\alpha}}{\partial x^i}A^{ji} \\ &= \frac{\partial x'^{\alpha}}{\partial x^i}\frac{\partial x'^{\beta}}{\partial x^j}A^{ij} \\ &= A'^{\alpha\beta} \end{aligned}$$

Therefore, symmetricity is preserved under coordinate transformation. □

(iii) Anti-symmetricity is also preserved under general coordinate transformation.

Proof. Let A^{ij} be a second rank contravariant tensor, and anti-symmetric, *i.e,*

$$A'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{i}} \frac{\partial x'^{\beta}}{\partial x^{j}} A^{ij}$$

and $A^{ij} = -A^{ji}$. Now

$$A'^{\beta\alpha} = \frac{\partial x'^{\beta}}{\partial x^{m}} \frac{\partial x'^{\alpha}}{\partial x^{n}} A^{mn}$$

Let us change m to j and n to i, then we have

$$\begin{aligned} A'^{\beta\alpha} &= \frac{\partial x'^{\beta}}{\partial x^{j}} \frac{\partial x'^{\alpha}}{\partial x^{i}} A^{ji} \\ &= -\frac{\partial x'^{\alpha}}{\partial x^{i}} \frac{\partial x'^{\beta}}{\partial x^{j}} A^{ij} \\ &= -A'^{\alpha\beta} \end{aligned}$$

Therefore, anti-symmetricity is also preserved under coordinate transformation. □

2.4.3. Covariant Metric Tensor

The quantities g_{ik} transform as a covariant tensor of rank 2[1].

Proof. We have

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^{\mu} dx^{\nu} \\ &= g_{\mu\nu} \left(\frac{\partial x^{\mu}}{\partial x'^{\alpha}} dx'^{\alpha} \right) \left(\frac{\partial x^{\nu}}{\partial x'^{\beta}} dx'^{\beta} \right) \\ &= \left(g_{\mu\nu} \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \right) dx'^{\alpha} dx'^{\beta} \\ &= g'_{\alpha\beta} dx'^{\alpha} dx'^{\beta} \end{aligned}$$

where

$$g'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu},$$

which is a covariant tensor of rank 2. □

[1] *This result follows from the assumption that ds^2 is invariant.*

3. Summary

Thus, from above discussion we have seen that how tensor plays an important role in physics as well as in mathematics. We have also seen different types of tensors, some special tensors, their properties, relevant proofs etc. We end this article here itself, with a note that the vectors and tensors described here are definable in *any* coordinate frame. Thus, we are not restricted to inertial frames or to linear transformations between such frames. Clearly this machinery will be useful to us in general relativity, where physics and dynamics are described in any general reference frame. Interested readers may go through it in details, if they want so, in any tensor calculus book, or in some relativity books.

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