Multiplicity Free Restrictions of Symmetric Groups

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March 12, 2014

Abstract

Using Gelfand's lemmas, I will be showing that the restriction of an irreducible representation of $\mathcal{S}_n$, when restricted to $\mathcal{S}_{n-1}$ is multiplicity free.

Introduction: Basics about Group Representation Theory

In this section, G denotes a finite group. I will mention a few definitions and the statement of the main theorem.

Definition 0.1. *Let G be a finite group and let V be a vector space over a field k(k could be a finite field, $\mathbb{R}$ or $\mathbb{C}$). Then a representation (V, σ) of G is defined as a homomorphism*

$$\sigma : G \to GL(V)$$

where $GL(V)$ denotes the group of invertible linear maps on V.

Definition 0.2. *Let (V, σ) be a representation of G. Let W be a subspace of V. We say that W is a G-invariant subspace of (V, σ) if for any $g \in G$, and any $w \in W$, $\sigma(g)(w) \in W$. Hence $\sigma(g)|_W$ is an ivertible linear map on W for all $g \in G$. Denoting $\sigma(g)_W$ by $\sigma_W(g)$, we have a representation (σ_W, W) of G and this is called a* ***sub-representation*** *of G.*

Here is one theorem which is very useful:

Theorem 0.3. ***Maschke's Theorem****: Let (V, σ) be a representation of a group G. Let W be a G-invariant subspace of W i.e., for any $g \in G$ and any $w \in W$, we have $\sigma(g)(w) \in W$. Then W has a G-invariant complement.*

Its proof can be found in the book Linear Representations of Finite Groups written by J. P. Serre.

Definition 0.4. *:**Invariant Vector**: Let (V, σ) be a finite dimensional representation of a finite group G. We say that a vector v is an <u>Invariant Vector</u> if $\sigma(g)v = v$ for all $g \in G$.*

The set V^G of invariant vectors of V is a subspace of V.

Definition 0.5. *Let (V, σ) be a representation of G. Suppose $V = \bigoplus_{\rho \in J} m_\rho W_\rho$(where $J \subseteq \hat{G}$) is the decompostion of V in to a direct sum of irreducible representations (W_ρ, ρ) of G. Then the direct sum of the m_ρ copies of W_ρ, $m_\rho W_\rho$ is called the ρ-**Isotypic Component** of (V, σ).*

Definition 0.6. *Let $(V, sigma)$ and (W, ρ) be representations of the group G. Then a linear transformation $T : V \to W$ is called an **intertwiner** if $T \circ \sigma(g)(v) = \rho(g) \circ T(v)$ for all $v \in V$ and for all $g \in G$.*

Definition 0.7. *:**Commutant**: Let (V, σ) be a representation of the group G. Then the space*

$$Hom_G(V, V) = \{T : V \to V \mid T \text{ is linear and } T \circ \sigma(g) = \sigma(g) \circ T, \ \forall\ g \in G\}$$

is called the commutant of (V, σ).

We have a representation (V, σ) of group G and $(V, \sigma) = \bigoplus_{\rho \in J} m_\rho(W_\rho, \rho)$. The ρ-isotypic component $m_\rho W_\rho$ is a direct sum of m_ρ orthogonal copies of W_ρ.
We have the following lemma:

Lemma 0.8. *The representation (σ, V) of group G is multiplicity-free if and only if its commutant $Hom_G(V, V)$ is a commutative algebra.*

Proof. We already know that if $(\sigma, V) = \bigoplus_{\rho \in J} m_\rho(\rho, W_\rho)$, where $J \subseteq \hat{G}$, then $Hom_G(V, V) = \bigoplus_{\rho \in J} M(m_\rho, \mathbb{C})$ with component wise multiplication.
Proof of $\Rightarrow$) If (σ, V) is multiplicity-free, then $m_\rho = 1$ for all $\rho \in J$. Thus, $Hom_G(V, V) = \bigoplus_{\rho \in J} \mathbb{C}$, which clearly is a commutative algebra(with component-wise multiplication).
$\Leftarrow$): If $Hom_G(V, V)$ is a commutative algebra, then we have that $\bigoplus_{\rho \in J} M(m_\rho, \mathbb{C})$ is a commutatuve algebra. That means, for each $\rho \in J$, $M(m_\rho, \mathbb{C})$ is commutative. But, the algebra $M(m_\rho, \mathbb{C})$ is commutative if and only if $m_\rho = 1$. Thus $m_\rho = 1$ for all $\rho \in J$. Thus (σ, V) is multiplicity free. □

Let G be a finite group and let $\hat{G}$ denote the set of irreducible representations of G over $\mathbb{C}$. Given a finite dimensional representation $(V, \ \sigma)$ of G, we can write $(V, \ \sigma) = \bigoplus_{\rho \in J} m_\rho W_\rho$, where $J \subseteq \hat{G}$, by repeatedly applying Maschke's Theorem. Here m_ρ is called the **multiplicity** of the irreducible representation (ρ, W_ρ).

Definition 0.9. *We say that a representation (V, σ) is **multiplicity free** if in the direct sum decomposition of (V, σ), the multiplicity m_ρ of each irreducible representation (ρ, W_ρ) is 1.*

Infact Maschke's theorem ensures us that the (V, σ) is the direct sum of irreducible subspaces, which are mutually orthogonal(with respect to the inner product).

We have an idea of what irreducible representations of the symmetric groups $\mathcal{S}_n$ are for $n = 3, 4, 5$(from character theory). Using character theory, we can see that an irreducible representation say ρ of $\mathcal{S}_5$ when restricted to $\mathcal{S}_4$ is a direct sum of inquivalent irreducible representations of $\mathcal{S}_4$ i.e., each irreducible representation of $\mathcal{S}_4$ occurring in $\rho|_{\mathcal{S}_4}$, occurs exactly once. Similarly for irreducible representations of $\mathcal{S}_4$ restricted to $\mathcal{S}_3$.
We cannot use this approach for $\mathcal{S}_n$ for n in general. Okounkov and Vershik's approach using Gelfand's lemma was used in proving the following theorem:

Theorem 0.10. *An irreducible representation $\mathcal{S}_n$ when restricted to $\mathcal{S}_{n-1}$ for $n \geq 2$, is multiplicity free.*

In this talk, we shall use Gelfand's lemma to prove the statement.

1 Permutation Representations

Let $G \circlearrowright X$. Then consider $L(X) = \{f : X \to \mathbb{C}\}$. We know that this is a vector-space , with the dirac functions, δ_x for $x \in X$ forming its basis.

Definition 1.1. *Define $\lambda : G \to GL(L(X))$ as $\lambda(g)(f)(x) = f(g^{-1}x)$ for all $x \in X$ and for all $g \in G$. This λ is a representation and this λ is called the* ***Permutation Representation****.*

Consider $G \circlearrowright X \times X$, with $g.(x, y) := (gx, gy)$. Then we consider the algebra $L(X \times X)$ with multiplication defined as $F_1.F_2(x, y) = \sum_{z \in X} F_1(x, z)F_2(z, y)$ for any $F_1,\ F_2 \in L(X, X)$. Consider the permutation representation of G on $L(X \times X)$. Now we have an important lemma:

Lemma 1.2. *$Hom_G(L(X), L(X)) \cong L(X \times X)^G$.*

Proof. Define $T : L(X \times X) \to Hom(L(X), L(X))$ as

$$T(F)(f)(x) = \sum_{y \in X} F(x, y)f(y)$$

Clearly this map is a linear map. Suppose $T(F)(f) = 0$ for all $f \in L(X)$. Then $T(F)(\delta_y) = 0$ for all $y \in X$. But $T(F)(\delta_y)(x) = \sum_{z \in X} F(x, z)\delta_y(z) = F(x, y)$. Thus $F(x, y) = 0$ for all $x, y \in X$. Thus T is one-one and since $dimL(X \times X) = |X|^2 = dimHom(L(X), L(X))$, we get that this T is a bijection, therefore an isomorphism between $L(X \times X) \to Hom(L(X), L(X))$. Now we define the same map on $L(X \times X)^G$. So we need to check that $T(F) \in Hom_G(L(X), L(X))$.

$$\begin{aligned} T(F) \circ \lambda(g)(f)(x) &= \sum_{y \in X} F(x, y)\lambda(g)f(y) \\ &= \sum_{y \in X} F(x, y)f(g^{-1}y) \\ &= \sum_{y \in X} F(g^{-1}x, g^{-1}y)f(g^{-1}y) \quad \because \ F \in L(X \times X)^G \\ &= \sum_{y \in X} F(g^{-1}x, y)f(y) \\ &= T(F)(f)(g^{-1}x) \\ &= [\lambda(g) \circ T(F)(f)](x) \end{aligned}$$

for all $x \in X$ and $\forall\ g \in G$. Thus $T(F)(f) \in Hom_G(L(X), L(X))$.
Given $S \in Hom_G(L(X), L(X))$, we know that $\exists F_S \in L(X \times X)$ such that $T(F_S) = S$. We

claim that $F_S \in L(X \times X)^G$. We have

$$\lambda(g)T(F_S)(f)(x) = \sum_{y\in X} F_S(g^{-1}x, y)f(y)$$
$$= \sum_{y\in X} F_S(x, y)f(g^{-1}y) \ \because\ T \in Hom_G(L(X), L(X))$$
$$but$$
$$\sum_{y\in X} F_S(g^{-1}x, y)f(y) = \sum_{y\in X} F_S(g^{-1}x, g^{-1}y)f(g^{-1}y)$$
$$\therefore\ \sum_{y\in X} F_S(x, y)f(g^{-1}y) = \sum_{y\in X} F_S(g^{-1}x, g^{-1}y)f(g^{-1}y)$$
$$\Rightarrow \sum_{y\in X}(F_S(x, y) - F_S(g^{-1}x, g^{-1}y))f(g^{-1}y) = 0 \ \forall\ x \in X$$

Therefore $F_S(x, y) - F_S(g^{-1}x, g^{-1}y) = 0$ for all $x,\ y \in X$ and for all $g \in G$. Thus $F_S \in L(X \times X)^G$. This proves that $Hom_G(L(X), L(X))$ is isomorphic to $L(X \times X)^G$. □

2 Gelfand Pairs and Gelfand's Lemma

From now on we will assume that $G \circlearrowright X$ is a transitive action. Fix x_0 and let K be the stabilizer of x_0, then we know that $G/K \cong X$ as G-sets. Given, this G and its subgroup K, we define some functions called *bi-K-invariant* functions in $L(G)$ as

Definition 2.1. *Let $f \in L(G)$ and let K be a subgroup of G. We say that f is a* ***bi-K-invariant*** *function if*

$$f(k_1gk_2) = f(g)$$

for all $g \in G$ and $k_1,\ k_2 \in K$.

Define $L(K\backslash G/K)$ to be the subspace of of all bi-K-invariant functions on G. It can be easily checked that $L(K\backslash G/K)$ is a 2-sided ideal of $L(G)$.

Proposition 2.2. $L(X \times X)^G \cong L(K\backslash G/K)$

Proof. Define $\Phi : L(X \times X)^G \to L(G)$. Where

$$\Phi(F)(g) = \frac{1}{|K|}F(x_0, gx_0)$$

Claim: $\Phi(F) \in L(K\backslash G/K)$. Let $g \in G$ and $k_1, k_2 \in K$. Then

$$\Phi(F)(k_1gk_2) = \frac{1}{|K|}F(x_0, k_1gk_2x_0)$$
$$= \frac{1}{|K|}F(k_1x_0, k_1gx_0) \because K = G_{x_0}(\text{the stabilizer of } x_0),\ x_0 = k_1x_0 = k_2x_0$$
$$= \frac{1}{|K|}F(x_0, gx_0)$$

which is equal to $\Phi(F)(g)$. Therefore $\Phi(F) \in L(K\backslash G/K)$. Clearly Φ is linear. We prove next that this map an algebra homomorphism. Let $F,\ F' \in L(X \times X)^G$. Then

$$\begin{aligned}
\Phi(FF')(g) &= \frac{1}{|K|} FF'(x_0, gx_0) \\
&= \frac{1}{|K|} \sum_{y \in X} F(x_0, y) F'(y, gx_0) \\
&= \frac{1}{|K|^2} \sum_{h \in G} F(x_0, hx_0) F'(hx_0, gx_0) \because \text{each } y \text{ repeats } |K| \text{ times.} \\
&= \sum_{h \in G} \left(\frac{1}{|K|} F(x_0, hx_0) \right) \left(\frac{1}{|K|} F'(x_0, h^{-1} g x_0) \right) \\
&= \Phi(F) * \Phi(F')(g)
\end{aligned}$$

For all $g \in G$. Next, if $\Phi(F)$ is identically zero, we have $F(x_0, gx_0) = 0$ for all $g \in G$. Thus, for any $x, y \in G$, $x = g_x x_0$ and $y = g_y x_0$. So $F(x, y) = F(g_x x_0, g_y x_0) = F(x_0, g_x^{-1} g_y x_0) = 0$. Thus $F \equiv 0$, therefore Φ is injective. Now, let $f \in L(K\backslash G/K)$. Then for $x, y \in X$, where $x = g_x x_0$ and $y = g_y x_0$, define $F(x, y) = |K| f(g_x^{-1} g_y)$. Then it is clear that $F(gx, gy) = F(x, y)$ for all $g \in G$ and for all $x, y \in X$. Then $\frac{1}{|K|} F(x_0, gx_0) = |K| \frac{1}{|K|} f(g) = f(g)$ for all $g \in G$. Thus Φ is an isomorphism. □

Corollary 2.3. $Hom_G(L(X), L(X)) \cong L(X \times X)^G \cong L(K\backslash G/K)$

Definition 2.4. *Let G be a finite group and let K be its subgroup. Then the pair (G, K) is said to be a* ***Gelfand pair*** *if for every $g \in G$, $g^{-1} \in KgK$.*

Let $G \circlearrowright X$ transitively with K being the stabilizer of a fixed $x_0 \in X$, then we have the following theorem.

Theorem 2.5. *If (G, K) is a Gelfand pair, then the permutation representation $(\lambda, L(X))$ is multiplicity-free.*

Proof. : To show that $(\lambda, L(X))$ is multiplicity-free, we only need to show that $L(K\backslash G/K)$ is commutative and theorem follows from Corollary 2.3.
Observe that, for any $f \in L(K\backslash G/K)$, $f(g^{-1}) = f(k_1 g k_2) = f(g)$, since (G, K) is a Gelfand pair. Let $f_1,\ f_2 \in L(K\backslash G/K)$. Then

$$\begin{aligned}
f_1 * f_2(g) &= \sum_{h \in G} f_1(gh) f_2(h^{-1}) \\
&= \sum_{h \in G} f_1(h^{-1} g^{-1}) f_2(h) \\
&= \sum_{h \in G} f_2(h) f_1(h^{-1} k_1 g k_2) \text{ for some } k_1,\ k_2 \in K \\
&= \sum_{h \in G} f_2(h) f_1(h^{-1} k_1 g) \because 1, k_2 \in K \text{ and } f \text{ is bi-}K\text{-invariant} \\
&= \sum_{h \in G} f_2(k_1 h) f_1(h^{-1} g) \because \text{we replaced } h \text{ by } k_1^{-1} h \\
&= \sum_{h \in G} f_2(h) f_1(h^{-1} g) \\
&= f_2 * f_1(g)
\end{aligned}$$

for all $g \in G$. Thus $L(K\backslash G/K)$ is commutative. □

3 Multiplicity-Free Subgroups

Let G be a finite group and let $H \subseteq G$, be a subgroup. Define the action $G \times H \circlearrowright G$ as

$$(g, h).g_0 = gg_0h^{-1}$$

for $g, g_0 \in G$ and $h \in H$. Clearly this defines a transitive action and it is very easy to check that $\tilde{H} = \{(h, h) \ : \ h \in H\}$ is the stabilizer of 1_G. Now consider the permutation representation η of $G \times H$ on $L(G)$ i.e.,

$$\eta(g, h)(f)(g_0) = f(g^{-1}g_0h)$$

for all $g, g_0 \in G$ and for all $h \in H$.

Definition 3.1. *Let G be a finite group. A subgroup H of G is said to be a **Multiplicity-Free subgroup** if for every $\rho \in \hat{H}$ and for every $\sigma \in \hat{G}$, the multiplicity of ρ in $\sigma|_H$ is*

$$dim(Hom_H(\rho, \sigma|_H)) \leq 1$$

.

Now we have a theorem, which characterizes multiplicity-free subgroups:

Theorem 3.2. *Given G a finite group and its subgroup H, we have for any $(\rho, W) \in \hat{H}$ and any $(\sigma, V) \in \hat{G}$,*

$$Hom_{G\times H}(\sigma \boxtimes \rho, \eta) \cong Hom_H(\rho, \sigma^*|_H)$$

where $\sigma \boxtimes \rho$ is the external tensor product of $\sigma \in \hat{G}$ and $\rho \in \hat{H}$ and σ^* denotes the contragradient of σ.

Proof. We know that the permutation representation η of $G \times H$ on $L(G)$ is

$$\eta = Ind_{\tilde{H}}^{G\times H} \iota_{\tilde{H}}$$

where $\iota_{\tilde{H}}$ is the trivial representation of $\tilde{H}$. Then we have

$$\begin{aligned} Hom_{G\times H}(\sigma \boxtimes \rho, Ind_{\tilde{H}}^{G\times H} \iota_{\tilde{H}}) &= Hom_{\tilde{H}}(\sigma \boxtimes \rho|_{\tilde{H}}, \iota_{\tilde{H}}) \text{ [Frobenius Reciprocity Theorem]} \\ &= Hom_H((\sigma \otimes \rho)|_H, \iota_H) \\ &= Hom(\sigma|_H \otimes \rho, \mathbb{C})^H \\ &= Hom(\rho, Hom(\sigma, \mathbb{C}))^H \\ &= Hom_H(\rho, \sigma^*|_H) \end{aligned}$$

This tells us that the multiplicity of ρ in $\sigma^*|_H$ is equal to the multiplicity of $\sigma \boxtimes \rho$ in the permutation representation of $G \times H$ on $L(G)$. □

A very important consequence of this theorem is as follows

Corollary 3.3. *H is a multiplicity-free subgroup of G if and only if the permutation representation representation of $G \times H$ on $L(G)$ is multiplicity-free.*

Corollary 3.4. *If $(G \times H, \tilde{H})$ is a Gelfand pair, then H is a multiplicity-free subgroup of G.*

Proof. H is multiplicity free if and only if the permutation representation of $G \times H$ on $L(G)$ is multiplicity free(by Theorem 3.2). The permutation representation of $G \times H$ on $L(G)$ is multiplicity free if $(G \times H, \tilde{H})$ is a Gelfand pair(Theorem 2.5). □

Lemma 3.5. *: $(G\times H,\tilde{H})$ is a Gelfand pair if and only if for every $g\in G$, $\exists\ h\in H$ such that $g^{-1}=hgh^{-1}$.*

Proof. If $(G\times H,\tilde{H})$ is a Gelfand pair, then for any $g\in G$ $(g,1)^{-1}=(h_1,h_1)(g,1)(h_2,h_2)$ for some $h_1,h_2\in H$. Thus we get

$$(g^{-1},1)=(h_1gh_2,h_1h_2)$$

which implies that $h_2=h_1^{-1}$. Therefore $g^{-1}=h_1gh_1^{-1}$.
Conversely, if g is H-conjugate to its inverse, then for any $g\in G$ and $h\in H$, we have

$$g^{-1}h=h_1h^{-1}gh_1^{-1}$$

for some $h_1\in H$. Then let $h_1'=h_1h^{-1}$ and let $h_2'=h_1^{-1}h^{-1}$. Then

$$\begin{aligned} h_1'gh_2' &= g^{-1}\\ h_1'hh_2' &= h^{-1}\end{aligned}$$

Therefore $(h_1',h_1')(g,h)(h_2',h_2')=(g^{-1},h^{-1})$. This happens for all $(g,h)\in G\times H$. Thus $(G\times H,\tilde{H})$ is a Gelfand pair. □

Proof. (Proof of Theorem 0.10): From Corollary 3.4 of Theorem 3.2, it suffices to prove that $(\mathcal{S}_n\times\mathcal{S}_{n-1},\tilde{\mathcal{S}_{n-1}})$ is a Gelfand pair. Here $\mathcal{S}_{n-1}$ permutes $\{1,2,\ldots,n-1\}$. We know that any element $w\in\mathcal{S}_n$ can be written as product of disjoint cycles. i.e.,

$$w=(a_{11}\to\ldots\to a_{1\lambda_1}\to a_{11})\ldots(a_{k1}\to\ldots\to a_{k\lambda_k}\to a_{k1})$$

where

$$1\le\lambda_k\le\lambda_{k-1}\le\ldots\le\lambda_1$$
$$\textit{and}$$
$$\sum_{i=1}^{k}\lambda_i=n$$

$\lambda(w):=(\lambda_1,\ldots\lambda_k)$ is called the cycle decomposition type of w. Let $r(w)$ denote the length of the cycle in the cycle decomposition of w, that moves n. Then conjugation of w by $z\in\mathcal{S}_{n-1}$ looks like

$$zwz^{-1}=(z(a_{11})\to z(a_{12})\to\ldots\to z(a_{1\lambda_1})\to z(a_{11}))\ldots(n\to z(a_{t2})\to\ldots$$

We claim that two elements $w,\ w'\in\mathcal{S}_n$ are $\mathcal{S}_{n-1}$-conjugate if and only if $\lambda(w)=\lambda(w')$ and $r(w)=r(w')$. The proof is as follows.
If for $w,\ w'\in\mathcal{S}_n$, suppose $\lambda(w)=\lambda(w')$ and $r(w)=r(w')=r$. So we have

$$\begin{aligned} w&=(a_{11}\to a_{12}\to\ldots\to a_{1\lambda_1}\to a_{11})\ldots(n\to a_{t2}\ldots\to a_{t\lambda_r}\to n)\ldots\\ w'&=(a_{11}'\to a_{12}'\to\ldots\to a_{1\lambda_1}'\to a_{11}')\ldots(n\to a_{t2}'\ldots\to a_{t\lambda_r}'\to n)\ldots\end{aligned}$$

Then define $\theta\in\mathcal{S}_n$ as $\theta(a_{ij})=a_{ij}'$ for all $i:\ 1\le\ i\le\ k$ and $j:\ 1\le\ j\ \le\ \lambda_i$. Then we have $\theta(n)=n$. Thus $\theta\in\mathcal{S}_{n-1}$ and $\theta w\theta^{-1}=w'$.
The converse is trivial.
So for any $w\in\mathcal{S}_n$,

$$w^{-1}=(a_{11}\leftarrow a_{12}\leftarrow\ldots\leftarrow a_{1\lambda_1}\leftarrow a_{11})\ldots(n\leftarrow a_{t2}\ldots\leftarrow a_{t\lambda_r}\leftarrow n)\ldots$$

Thus w^{-1} is of the same cycle-type as w i.e., $\lambda(w^{-1})=\lambda(w)$ and $r(w^1)=r(w)$. Thus w and w^{-1} are $\mathcal{S}_{n-1}$-conjugate, therefore,$(\mathcal{S}_n\times\mathcal{S}_{n-1},\tilde{\mathcal{S}_{n-1}})$ is a Gelfand pair(by Lemma 3.5) and thus for every $\sigma\in\hat{\mathcal{S}}_n$, $\sigma|_{\mathcal{S}_{n-1}}$ is multiplicity-free. This is for every $n\ge 2$ □