Automorphisms of the Upper Half Plane

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Some Terminology

Let $\mathbb{C}$ be the set of all complex numbers. Let $\mathbb{H} \subset \mathbb{C}$ be such that

$$\mathbb{H} = \{z | \Im(z) > 0\}$$

$\mathbb{H}$ is an open subset of $\mathbb{C}$.

Given open sets $\mathcal{U}, \mathcal{V} \subset \mathbb{C}$, a function $f : \mathcal{U} \to \mathcal{V}$ is said to be biholomorphic if it is holomorphic on $\mathcal{U}$, bijective and its inverse $f^{-1} : \mathcal{V} \to \mathcal{U}$ is holomorphic on $\mathcal{V}$.

For an open subset $\mathcal{U}$, we define

$$Aut(\mathcal{U}) = \{f : \mathcal{U} \to \mathcal{U} : f \text{ is biholomorphic}\}$$

We note that $Aut(\mathcal{U})$ is has a group structure with the composition of biholomorphic functions as the group operation and the identity function as the unit element.

In this article we are interested in $Aut(\mathbb{H})$.

Action of Some Matrix Groups on the Upper Half Plane

We define action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

It is not difficult to see that the action is well defined.

For $A \in SL_2(\mathbb{R})$, we define the map $f_A : \mathbb{H} \to \mathbb{H}$,

$$f_A(z) = A \cdot z$$

Here, f_A is a holomorphic function for all $A \in SL_2(\mathbb{R})$. We make the following observations:

- $f_A \circ f_B = f_{AB}$ for all $A \in SL_2(\mathbb{R})$
- $f_I = id_{\mathbb{H}}$
- f_A is holomorphic for all $A \in SL_2(\mathbb{R})$

From these observations we conclude that f_A is biholomorphic for all $A \in SL_2(\mathbb{R})$. Thus, this action of $SL_2(\mathbb{C})$ induces a homomorphism Φ from $SL_2(\mathbb{R})$ to $Aut(\mathbb{H})$ viz.,

$$\Psi : SL_2(\mathbb{R}) \to Aut(\mathbb{H})$$

$$\Psi(A) = f_A$$

If $A \in Ker(\Psi)$, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then

$$A \cdot z = \frac{az + b}{cz + d} = z$$

This leads to the quadractic equation

$$cz^2 + (d - a)z - b = 0$$

As this equation is satisfied by all complex numbers in the upper half plane and as a quadratic equation cannot have more than two roots the above polynomial is zero. This leads to

$$Ker(\Psi) = \{I, -I\}$$

Hence this induces an injective homomorphism Φ from $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/Ker(\Psi)$ to $Aut(\mathbb{H})$.

$$PSL_2(\mathbb{R}) \xrightarrow{\Phi} Aut(\mathbb{H})$$

Mobius Transformation of the Upper Half Plane to the Unit Open Ball

Let $\tau : \mathbb{H} \to \mathbb{C}$ be the map:

$$\tau(z) = \frac{z - \iota}{z + \iota}$$

and let $\sigma : B \to \mathbb{C}$, where B is the unit open ball around the origin, be the following map:

$$\sigma(z) = \frac{z + 1}{\iota z - \iota}$$

We make the following observations :

- τ and σ are holomorphic.
- $|\tau(z)| = |\frac{z-\iota}{z+\iota}| < 1$.
- $\Im(\sigma(z)) = \Im(\frac{z+1}{\iota z-\iota}) > 0$.
  Thus, image of τ is B and that of σ is $\mathbb{H}$. With abuse of notation from here on we denote these two maps as the ones whose co-domains are their respective images.
- $\tau(\iota) = 0$, $\sigma(0) = \iota$.
- $\tau \circ \sigma(z) = z$ and $\sigma \circ \tau(z) = z$.

Thus, from the above observations we conclude that τ and σ are biholomorphic maps and are inverses of each other. Thus, the automorphism groups $Aut(B)$ and $Aut(\mathbb{H})$ are isomorphic. This is because, if $f \in Aut(\mathbb{H})$ then the map

$$f \mapsto \tau \circ f \circ \sigma$$

is an isomorphism of the two groups. Let us call this isomorphism as χ.

Automorphisms of B which fix the Origin

In this section we make use of the Schwarz lemma which we state without proving:

> If $f : B \to B$ is a non-constant holomorphic map such that $f(0) = 0$ then,
>
> $$|f(z)| \leq |z|$$
>
> If $|f(z)| = |z|$ for some $z \in B$ then
>
> $$f(z) = \lambda z$$
>
> where $|\lambda| = 1$.

If $f \in Aut(H)$ such that $f(0) = 0$ then,

$$|f(z)| \leq |z|$$

Also,

$$|f^{-1}(z)| \leq |z|$$

But that gives

$$|z| \leq |f(z)|$$

Thus, $|f(z)| = |z|$. Thus, by Schwarz lemma we have that

$$f(z) = \lambda z$$

for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

Hence, all automorphisms of B which fix the origin are rotations about the origin.

Stabiliser Subgroups of $Aut(\mathbb{H})$ and $Aut(PSL_2(\mathbb{R}))$ at ι

As $Aut(B)$ and $Aut(\mathbb{H})$ are isomorphic, and the image of 0 is ι, $Stab_{Aut(B)}(0)$ and $Stab_{Aut(\mathbb{H})}(\iota)$ are isomorphic under the isomorphism χ^{-1}.

If $g \in Stab_{Aut(B)}(0)$ then $g(z) = \lambda z$ where $|\lambda| = 1$. If $\sqrt{\lambda} = a + \iota b$ for some $a, b \in \mathbb{R}$ then we can see that for $T = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$,

$$f_T = \sigma \circ g \circ \tau = \chi^{-1}(g)$$

Thus, $Stab_{Aut(B)}(0)$ is isomorphic to a subgroup of $Stab_{PSL_2(\mathbb{R})}(\iota)$. But this gives that the subgroups $Stab_{Aut(B)}(0)$ and $Stab_{PSL_2\mathbb{R}}(\iota)$ are isomorphic.

Automorphism Groups of the Open Unit Ball and the Upper Half Plane

It is easy to see that $Aut(\mathbb{H})$ and $PSL_2(\mathbb{R})$ have a transitive action on $\mathbb{H}$. Infact, given $x + iy \in \mathbb{H}$ we have

$$T = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}$$

and that

$$A.\iota = x + \iota y$$

Thus, $PSL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$. As it is isomorphic to a subgroup of $Aut(\mathbb{H})$, $Aut(\mathbb{H})$ also acts transitively on $\mathbb{H}$.

In the previous section we established that $Stab_{Aut(\mathbb{H})}(\iota)$ and $Stab_{PSL_2(\mathbb{R})}(\iota)$ are isomorphic. For $g \in Aut(\mathbb{H})$ let

$$\tilde{g} = g(\iota)$$

As $PSL_2(\mathbb{R})$ acts transitively, there exists $h \in PSL_2(\mathbb{R})$ such that

$$h.\tilde{g} = \iota$$

Hence, $\Phi(h)g(i) = i$ and hence, $\Phi(h)g$ fixes ι. Thus, $\Phi(h)g$ has a unique pre-image in $Stab_{PSL_2(\mathbb{R})}(\iota)$ and hence g has a unique pre-image in $PSL_2(\mathbb{R})$.

Thus, we see that Φ is an isomorphism. Hence, the automorphism groups of the unit open ball and the upper half plane are both isomorphic to the projective special linear group $PSL_2(\mathbb{R})$.