Mathematics is fun and games!

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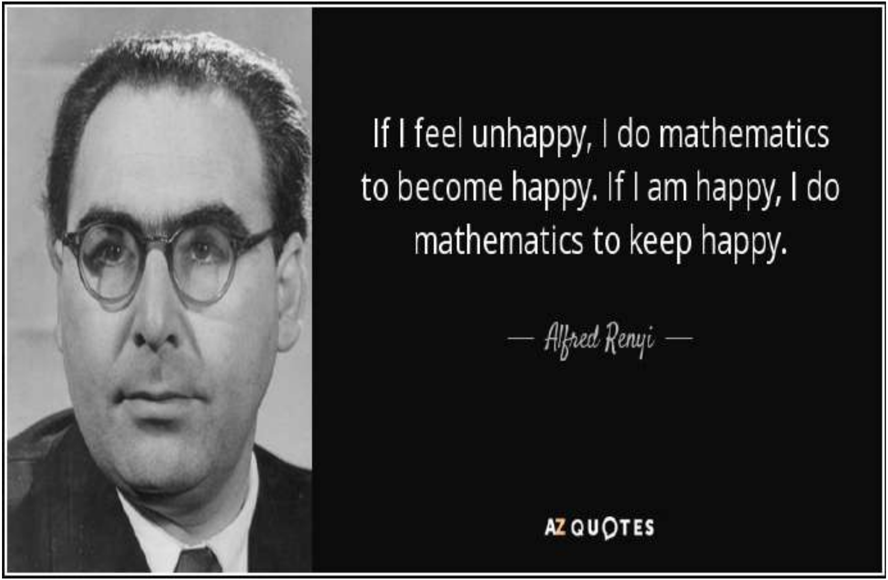
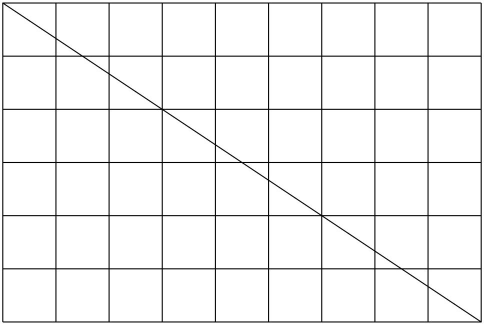

If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy.
— Alfred Renyi —
AZ QUOTES

"Ein Mathematiker ist eine Maschine, die Kaffee in Satze verwandelt."

Rectangular grid

Draw a rectangular grid made up of unit squares; for example, a rectangle of length 9 units and height 6 units.
Draw a diagonal - say, from the left upper corner to the lower right corner.
How many of the small unit squares does this diagonal pass through?



What about a 42×18 rectangle? Or a general $m \times n$ rectangle?!

For 9×6, we got 12.

For 42×18, we will get 54.

For a general $m \times n$ rectangle, we will get...?

So, the answer is $m + n - GCD(m, n)$ which tells us we have a geometric way of realizing the GCD!

We saw that the GCD can be calculated geometrically; here is one more example of this friendship of geometry and number theory.

For coprime integers $m, n > 1$ look at the $m \times n$ rectangle filled as follows:

Start with the left top corner and start filling the digits 1,2,.. etc. diagonally downwards until you hit an edge when you 'fold' the rectangle and continue!

For instance, for 4×5, it looks like:

1	17	13	9	5
6	2	18	14	10
11	7	3	19	15
16	12	8	4	20

In particular, not only is every square filled by this process, but the (i, j)-th entry is:

the unique integer $< mn$ that leaves remainder i when divided by m and remainder j when divided by n.

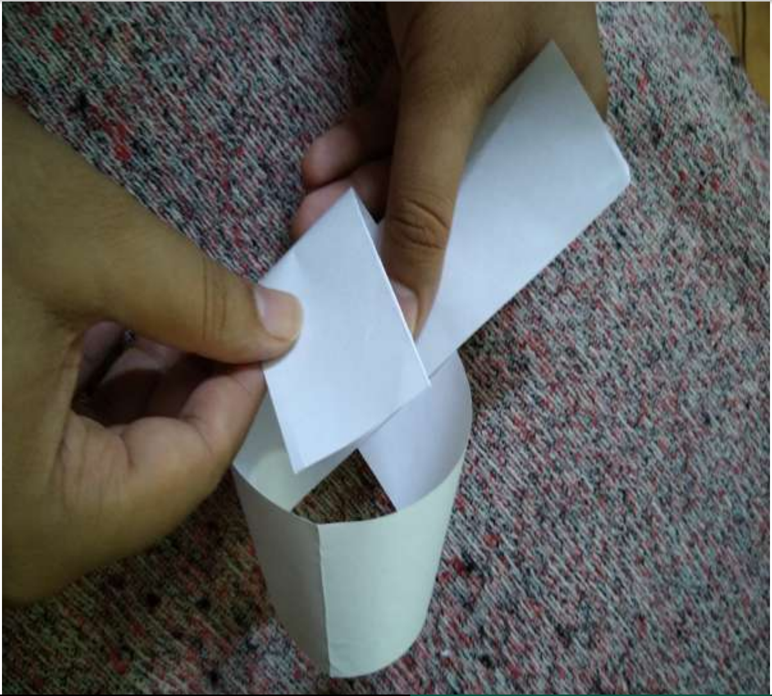
Thus, we can simply read out the solutions to the Sun-Tsu theorem by this process!

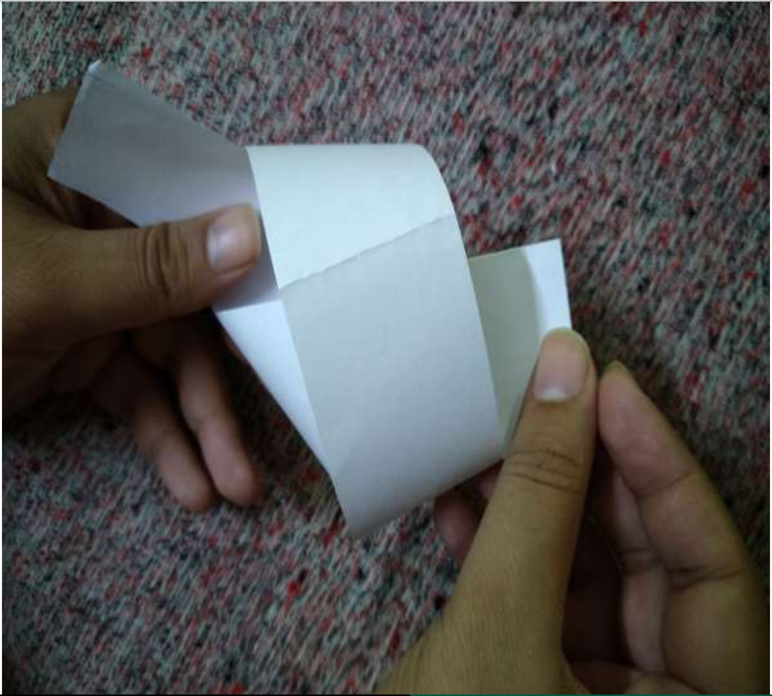
Take a thin rectangular piece of paper.

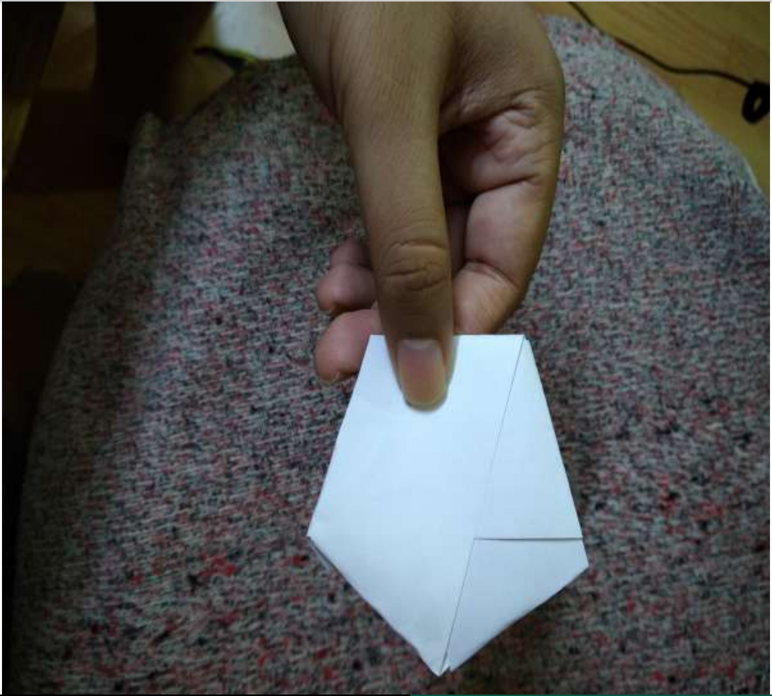
Make a knot without crumpling or tearing the paper.

Flatten the paper and fold or snip off the extra parts sticking out from the knot.

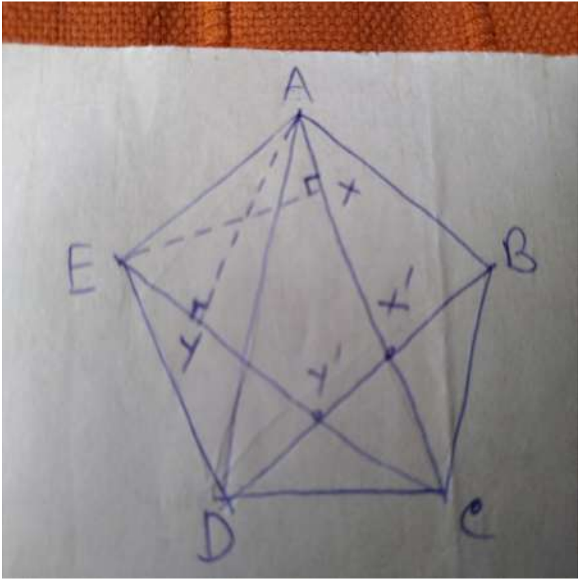
What is the shape we get?







The proof is simple 9th class geometry.



As $ACDE$ is simply the paper wrapped up, $EX = AY$.
$EX \perp AC, AY \perp CE$, the area of ACE equals
$\frac{1}{2}AY.CE = \frac{1}{2}EX.AC$; hence $CE = AC$.
Also $AE||BD$ as the paper is rectangular; hence

$$\frac{CX'}{AC} = \frac{CY'}{CE}$$

Hence $CX' = CY'$ and so $EY' = AX'$.
But $ABY'E$ is a parallelogram, $EY' = AB$.
Similarly, $AX' = ED$, $AB = DE$, $AE = BC$, $AD = AC, CE = AC$.
Thus, $AD = CE$.

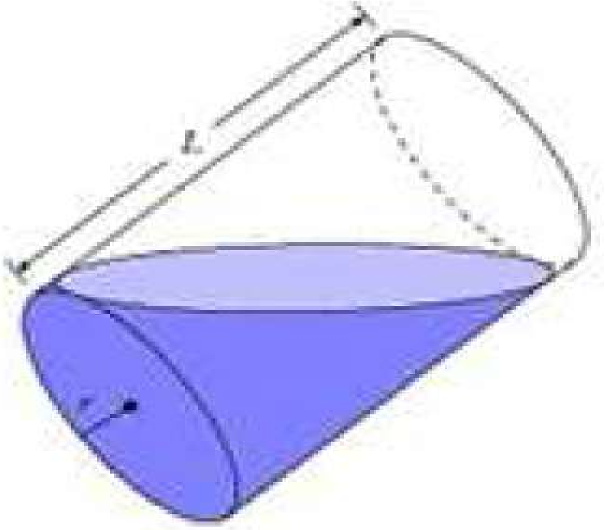
$DE||AC$ means $ACDE$ is a trapezium with equal diagonals, and hence $AE = CD, AE = BC = CD, AB = DE$.

Triangles AED and CBA are congruent by SSS.

Hence the angle $EDA = CAB$ but $EDA = DAC$ since $DE||AC$.
As $AD||BC$, angle $DAC = ACB$.
These give angle $ACB = CAB$ which gives $AB = BC$ and hence the pentagon is regular.

Here is a problem on filling glasses with water.

How do we know that we have filled a glass exactly half?



We can always check if two glasses have equal levels by placing them side by side.

What proportions are fillable in a finite number of steps and is there an algorithm to do this?

For example, clearly we can fill $(1/4)$-th, $(1/8)$-th etc. but can we fill $(3/8)$-th for instance?

In fact, follow the process : Fill left one, level both, fill left, level both, empty right, level both. These can be symbolically said as F,L,F,L,E,L.

The sequence FLFLFL respectively gives the following levels on the left and right side glasses

$$(1, 0), (1/2, 1/2), (1, 1/2), (3/4, 3/4), (0, 3/4), (3/8, 3/8).$$

Note that we don't need LL, FF, EE, FE, EF and the string consists of FL's and EL's always.

Note that $3 = (11)_2$ in binary notation and $8 = (1000)_2$ which means the fraction $3/8$ equals 0.011.

Reversing the idea, the procedure can be read off from the binary number 0.011 simply by reversing to 110 and calling FL as 1 and EL as 0 (reversing is needed because we write the procedure FL, FL,EL from left to right).

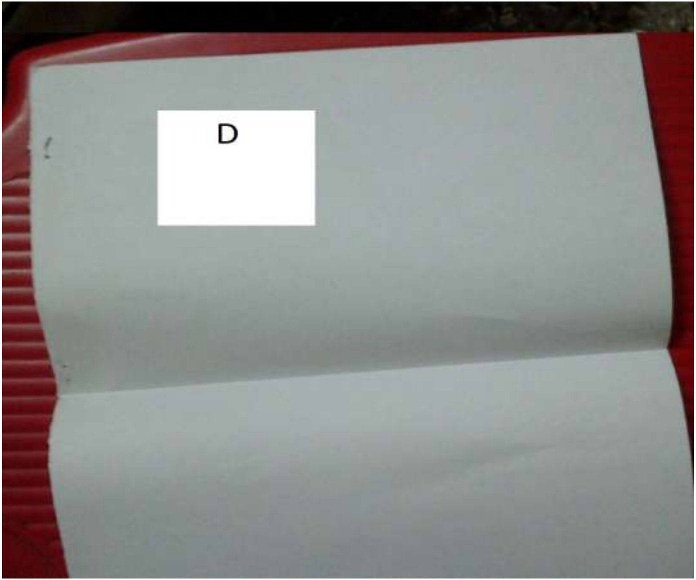
For example $47/64 = 0.101111$ which means the procedure

$$FL, FL, FL, FL, EL, FL$$

corresponding to 111101 will lead to filling (47/64)-th of a pot.

Folds

Take a rectangular piece of paper and make folds as follows.
First, fold the paper into half by bringing the bottom edge above to match with the top edge.
On this folded sheet, perform the same operation - that is, fold into half by bringing the bottom edge on top to match the top edge.
If you do this a number of times, say 10, times, and unfold the paper, there will be crests and troughs (ups and downs) on the papers.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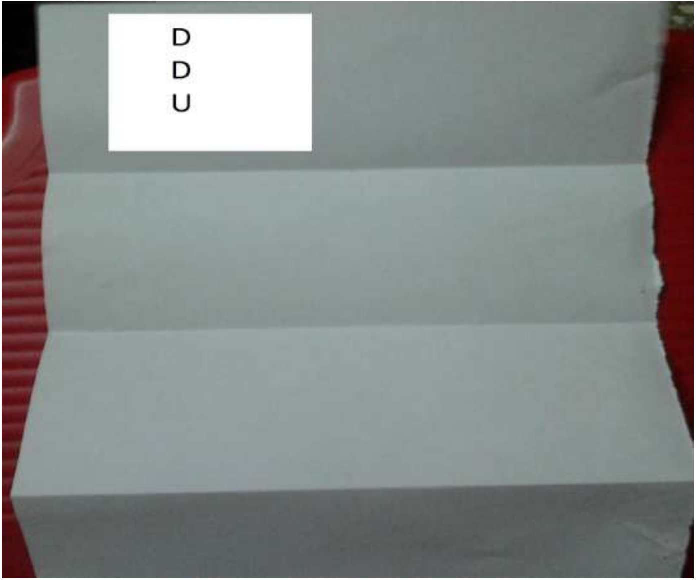
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Easy Question : What is the number of creases for n folds?

Observing from top to bottom, what is the pattern of 'ups' and 'downs'?

Does the pattern for $n+1$ folds start with the pattern of n folds?

We can see that the $2r$-th crease is the same as the r-th crease; in fact, writing $1, -1$ for the down and up, we have

$$f_{2^r(2n+1)} = (-1)^n f_{2^r}.$$

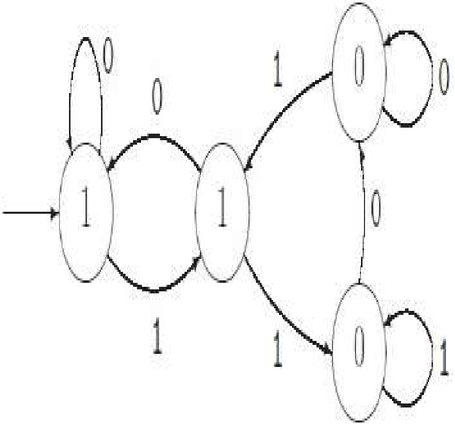
In fact, we can do folding upwards or downwards at each stage and this creates many paper-folding sequences.

Although we can make many observations (for instance, a paper-folding sequence unfolded any number of times gives a paper-folding sequence!), the sequences are still mysterious.

For instance, it is never eventually periodic!

The series $\sum_n f_n x^n$ evaluated at any algebraic number x strictly between 0 and 1 gives a transcendental value! (hence eventual periodicity is ruled out).

The sequence is 'automatic'; it is produced by the 2-automaton below. I don't say any more on this but feel free to look for 'folding sequence' with the help of Sundar Pichai.



The generating 2–automaton of the regular paperfolding sequence.

If you can't beat them, join them

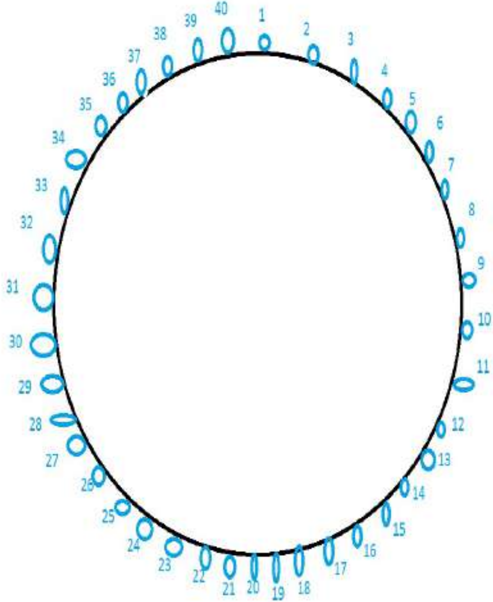
Story of Josephus:

Flavius Josephus and 39 of his comrades were surrounded when holding a revolt against the Romans during the 1st century A.D.
Rather than become slaves, they decided to kill themselves.
They arranged themselves along a circle.
Starting somewhere, they went clockwise around the circle and every 7th person was eliminated. This continued with the 7th among the surviving ones being killed at each step.
Apparently, Josephus was a clever mathematician and arranged himself in such a position that he would be the last survivor.
The story goes that he did not kill himself but came and joined the Romans!
We need to find out Josephus's position.

The general problem is of n people, designated by $1, 2, \cdots, n$ in clockwise order, say.

Each d-th person is eliminated going around in the clockwise direction.

What is the position of the last survivor?



FIRST FEW
CASUALTIES:

7, 14, 21, 28, 35, 2, 10, 18, 26, 34, 3, 12, 22, ...

This is a problem about permutations!
If a_r is the r-th person to be eliminated, we have a permutation $a_1, a_2, \cdots, a_n$ of the numbers $1, 2, \cdots, n$.
The last survivor is a_n;
There is no known nice closed formula for a_n in terms of n and d in general although the permutation itself can be described explicitly!
Fortunately, we can find a formula when every second person is killed; that is, when $d = 2$.

This formula can be neatly expressed in binary digits, viz.
If $n = d_r d_{r-1} \cdots d_1 d_0$ is the binary expansion of n, then assuming that we start with eliminating 2, then 4 etc., we have:
$d_{r-1} d_{r-2} \cdots d_1 d_0 d_r$ as the binary digit representing the position of the last person to be killed!
For example, if the number of people is $26 = (11010)_2$, then the person to be killed last is $(10101)_2 = 21$.
So, if A to Z are sitting cyclically and every second person is killed, then U remain(s) until the end!

To explain the above solution, let us rephrase the problem as follows.

A pack of n cards is disposed off as follows. The top card is placed at the bottom, the next card discarded, the next top card is placed at the bottom and the next card is discarded etc. and the process continued until only one card is left

Let $f(n)$ denote the position of this last (called selected') card originally from the top.

We first note that if $f(n) = 1$, then the number of cards must be some 2^r. For, if we reverse the process, every card brought from bottom to top must first be brought in. So, when our selected card completes a full cycle, the number of cards is doubled.

Now, let the number n of cards be in $[2^r, 2^{r+1})$.

For the selected card to come to the top, as we saw above, we must bring down the number of cards from n to 2^r which means discarding $n - 2^r$ cards.

While discarding $n - 2^r$ cards, the same number of cards have gone to the bottom.

Hence, as $2(n-2^r)$ cards have been taken from the top of the selected card, its position is

$f(n) = 2(n-2^r)+1 = 2n-2^{r+1}+1.$

Since r is the greatest integer not exceeding $\log_2(n)$ (denoted $[\log_2(n)]$), we have $f(n) = 2n - 2^{[\log_2(n)]+1} + 1$; in terms of binary expansions, $n = a_1 a_2 \cdots a_d$ means $f(n) = a_2 a_3 \cdots a_d a_1$.

In general, among n people with people getting eliminated as $d, 2d$ etc., the r-th person to be eliminated is

$$(1, 2, \cdots, n)^{d-1}(2, 3, \cdots, n)^{d-1} \cdots (n-1, n)^{d-1}(r)$$

where we read from right to left.

Stable table

Here is a problem which can be solved using high school or perhaps beginning college level mathematics:

If you are seated at a wobbly rectangular table in a restaurant, what would you do to make the table steady?

Assume the floor is continuous and its inclination is not more than 35.26 degrees - this is the smallest value at which the value of tan function becomes more than $1/\sqrt{2}$.

Assume that the four feet of the table have lengths at least half the diagonal of the table.

Without any leg digging into the floor, the table can simply be rotated to get to a stable position!

This is a consequence of the IMVT.

Here is a trick that the magician in you can perform with an associate (we used this in the Regional Math Olympiad in Delhi).

A deck of 52 cards is given. There are four suites each having cards numbered 1 to 13. The audience chooses some five cards with distinct numbers written on them. The associate of the magician comes by, looks at the five cards and turns exactly one of them face down and arranges all five cards in some order. Then the magician enters and with an agreement made beforehand with the associate, she determines the face down card - both suite and number without touching anything!

The solution is:

The associate performs the following trick.
She chooses two cards of the same suite, and turns one of them face down and arranges the face down and face up card in a way so that the smaller card is to the left (it could be face down or face up).

Next, she takes the remaining three cards and puts them to the right of these two and arranges them in the following way.

Since the three have three distinct numbers on them, there are SIX ways to arrange them.

In an agreement between both of you made beforehand, each of these arrangements corresponds to a number between 1 and 6 since only the order of the cards matter.

You enter and see that the face down card is either to the left most or second from the left.

In both cases, you know the suite of the face down card and also know whether it is smaller or bigger according to whether it is the 1st or the 2nd from the left.

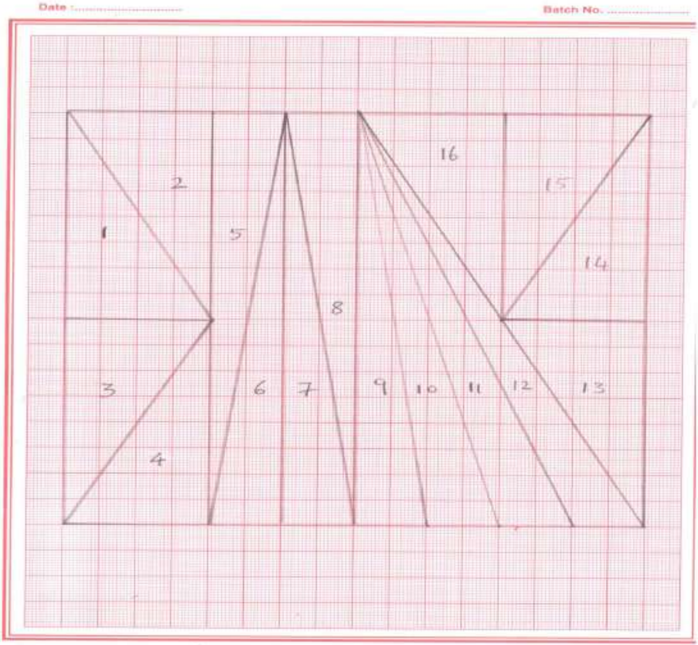
Looking at the third, fourth and fifth card from the left, and the agreement on numbers, she can get a number between 1 and 6.

The associate arranges them so that this number is precisely the distance between the face down card and the card of the same suite when all the thirteen cards are arranged on a circle.

Since you also know whether the face down card has a larger number from the chosen face up neighbour of the same suite, and what the difference between them on a circle is, Voila!

Tiling squares by triangles of given area

Try to cut a square into finitely many triangles (possibly of different shapes) of equal area.
You would find that - no matter what the shapes are - the number of triangles is always even!
We shall discuss a proof of this shortly. Amazingly, it uses some nontrivial mathematical objects called 2-adic valuations!



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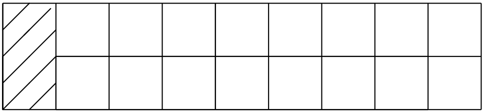
An old problem arising in Statistical Mechanics involves dominos; a domino is a rectangle formed by joining together two unit squares along an edge.

One tiles a rectangular grid (thought of as a lattice) using dominos (thought of as two molecules connected by a bond) and uses this as a model for molecules on a lattice.

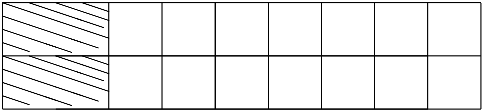
From the number of domino tilings, thermodynamical properties can be calculated when there is a so-called zero energy of mixing.

For a $2 \times n$ grid, can we determine the number c_n of domino tilings?

The only choices are: we can either place a single vertical domino on the first column:



or two horizontal dominos in the two left-most columns:



This easily gives the recursion $c_n = c_{n-1} + c_{n-2}$ and, looking at the small cases of n, gives us ......?

??

Yes, the Fibonacci sequence!

The $2 \times n$ grid was easy to deal with but for more general grids, one attaches a so-called bipartite graph to each tiling and studies the problem through graph theory.

Physicists Kasteleyn and, independently, Temperley and Fisher found the following astounding formula for the number of domino tilings of an $m \times n$ grid where m is even:

$$\prod_{r=1}^{m/2} \prod_{s=1}^{n} 2\sqrt{\cos^2\left(\frac{r\pi}{m+1}\right) + \cos^2\left(\frac{s\pi}{n+1}\right)}.$$

Towards the proof of the result on evenness of the number of triangles, consider the "2-adic valuation map" $\phi : \mathbf{Q}^* \to \mathbf{Q}$ defined as follows:
Write any $x = \frac{2^a b}{c}$) in $\mathbb{Q}^*$ where b, c are odd and a could be 0, positive or negative; define $\phi(x) = a$.
Define also $\phi(0) = \infty$.

Colour a point $(x, y) \in \mathbf{Q} \times \mathbf{Q}$ by three colours as follows:

red, if $\phi(x), \phi(y) > 0$,

blue, if $\phi(x) \leq 0; \phi(x) \leq \phi(y)$,

green, if $\phi(x) > \phi(y)$ and $\phi(y) \leq 0$.

(x, y) is red if the numerators are both even,

(x, y) is blue if $x = b/2^u c$ with b, c odd and $u \geq 0$ and, either the numerator of y is even or $y = b'/2^v c'$ with b', c' odd and $v < u$.

(x, y) is green if $y = b'/2^v c'$ with b', c' odd and $v \geq 0$ and either the numerator of x is even or $x = b/2^u c$ with b, c odd and $u < v$.

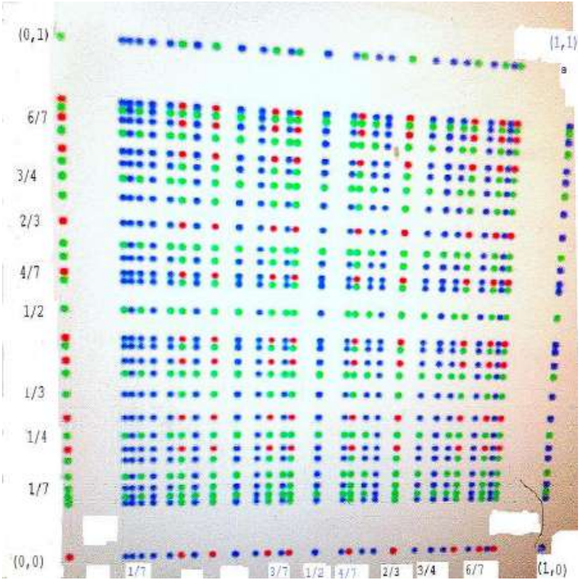
$(2, 0)$ is red, $(1, 3)$ is blue and $(1, 1/2)$ is green.

$(0, 0)$ is red while $(1, 0)$ is blue and $(0, 1)$ is green.

It is possible to extend this function to a function

$$\phi : \mathbf{R} \to \mathbf{R}$$

which satisfies :
ϕ restricts to the 2-adic valuation on $\mathbf{Q}$;
$\phi(xy) = \phi(x) + \phi(y)$;
$\phi(x + y) \geq min(\phi(x), \phi(y))$.
This is important to have such an extension because we would really like to colour all points in a square!
For instance, $\phi(\sqrt{3}/2) = -1$.


(0,1)
(1,1)
6/7
3/4
2/3
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1/2
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(0,0)
1/7
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1/2
4/7
2/3
3/4
6/7
(1,0)

We may take the unit square with a left lower corner at $(0, 0)$.
Here are a few easy observations :
(i) *If a point* a *is red, then any point* x *and* $x + a$ *have the same colour.*
(ii) *On any line, there are at the most two colours.*
(iii) *The boundary of the square has an odd number of segments which have a red end and a blue end.*
(iv) *If a triangle is not 'complete' (that is, has vertices only of one or two colours), then it has* 0 *or* 2 *red-blue edges.*

To prove these, recall

$$\phi(xy) = \phi(x) + \phi(y)$$

$$\phi(x + y) \geq min(\phi(x), \phi(y))$$

In particular, if $\phi(x) > \phi(y)$, then $\phi(x + y) = \phi(y)$.
In particular, if (x, y) is blue, then $\phi(x) \leq \phi(y)$ and so, $\phi(y/x) \geq 0$ and, if (x, y) is green, then $\phi(x) > \phi(y)$ and so, $\phi(y/x) < 0$.

Let us prove (i) now.
As a is red, its co-ordinates have positive ϕ and it is easy to check in each of the three cases of colouring for a point x that x and $x + a$ have the same colour.

For (ii), without loss of generality, we may assume that the line passes through the origin. But two other points $(x_i, y_i); i = 1, 2$ on the line $y = tx$ have colours blue and green respectively, say.
But then $\phi(y_1/x_1) = \phi(y_2/x_2) = \phi(t)$ is impossible as the former is ≥ 0 while the latter is < 0.

To prove (iii), note that (ii) implies that such segments on the boundary must be on the segment from $(0, 0)$ to $(1, 0)$ which are red and blue respectively. But then it is clear.
The proof of (iv) is completely clear by considering the various possibilities
RRB,RBB,RGG,RRG,BBG,BGG.
We have the even-ness theorem as follows.
Counting the red-blue edges on the square, we are counting the interior edges twice and the boundary edges once.
Thus, (iii) would be contradicted unless there is a complete triangle. But then a complete triangle has area A with $\phi(A) < 0$ - this can be checked (see the next slide) using the expression of the area A as a determinant and is the main point of introducing a colouring.

The main observation is then that there is ALWAYS a complete triangle; we show that a complete triangle has area A which satisfies $\phi(A) < 0$.
This would complete the proof because if there are n triangles, each of same area A, then $A = 1/n$ and the assertion $\phi(A) = \phi(1/n) < 0$ means n is even!
Firstly, note that the triangle can be moved so that the vertices are at $(0,0), (a,b)$ and (c,d) where (a,b) is blue and (c,d) is green. Thus, the area is $(ad - bc)/2$.
As (a,b) is blue, $\phi(a) \leq \phi(b)$ and as (c,d) is green, $\phi(c) > \phi(d)$.
Therefore, $\phi(ad) = \phi(a) + \phi(d) < \phi(b) + \phi(c) = \phi(bc)$ which gives $\phi(ad - bc) = \phi(ad) = \phi(a) + \phi(d) \leq 0$.
Hence $\phi(A) = \phi((ad - bc)/2) \leq -1$.

Tiling rectangles by rectangles

Let us discuss tiling integer rectangles with integer rectangles now. Can we tile a rectangle of size 28×17 by rectangles of size 4×7 ? At least the area of the smaller rectangle divides that of the larger one. But, in fact, we don't have a tiling! Why?
Look at each row of the big rectangle. If we have managed to tile as required, then 17 would be a positive linear combination of 4 and 7. This is impossible!

Thus, a necessary condition for tiling an $m \times n$ rectangle with $a \times b$ rectangles is that ab should divide mn and each of m, n should be expressible as positive linear combinations of a, b.
Is this condition sufficient ?
Look at a 10×15 rectangle which we wish to tile with copies of a 1×6 rectangle.
The two necessary conditions mentioned clearly hold true.
However, this tiling is not possible !

In fact, we claim that for an $a \times b$ rectangle to tile an $m \times n$ rectangle, it is also necessary that a must divide either m or n and b also must divide m or n.
To see why, look at a possible tiling.
We may suppose $a > 1$ (if $a = b = 1$, there is nothing to prove).
We colour the unit squares of the $m \times n$ rectangle with the different a-th roots of unity $1, \zeta, \zeta^2, \cdots, \zeta^{a-1}$ as follows.

1	z	z^2	..	1/z	1	z	..	
z	z^2	..	1/z	1	z	..		
z^2	..							
..								
1/z								
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z								
..								

$a(i,j)=z^{i+j-2}$

Think of the rectangle as an $m \times n$ matrix of unit squares and colour the (i, j)-th unit square by ζ^{i+j-2}.
Since each tile (copy of the smaller rectangle used) contains all the a-th roots of unity exactly once, and as the sum $1 + \zeta + \cdots + \cdots \zeta^{a-1} = 0$, the sum of all the entries of the $m \times n$ rectangle must be 0.
Therefore, $\sum_{i=1}^{m} \sum_{j=1}^{n} \zeta^{i+j-2} = 0$.
But this sum is $(\sum_{i=1}^{m}, \zeta^{i-1})(\sum_{j=1}^{n} \zeta^{j-1}) = 0$ which means one of these two sums must be 0.
But $\sum_{i=1}^{m} \zeta^{i-1} = 0$ if, and only if, $\zeta^{m} - 1 = 0$; that is, $a|m$. The other sum is 0 if, and only if, $a|n$.
Thus, this condition that a divides m or n is necessary and, by the same reasoning it is necessary for tiling that b divides m or n.

Looking at the above proof, it is also easy to see how to tile when these conditions hold good. That is, we have:

An $m \times n$ rectangle can be tiled with $a \times b$ rectangles if, and only if,
(i) ab divides mn,
(ii) m and n are expressible as non-negative linear combinations of a and b,
(iii) a divides m or n and b divides m or n.

This generalizes in an obvious way to any dimension !

We discuss now the following result for which several proofs are available!

If a rectangle is tiled by rectangles each of which has at least one of its sides integral, then the big rectangle must also have a side of integral length.

We place the co-ordinate system such that all the sides of the rectangles have sides parallel to the co-ordinate axes.

Consider the function $f(x, y) = e^{2i\pi(x+y)}$ for $(x, y) \in \mathbf{R}^2$.
For a rectangle defined by $[a, b] \times [c, d]$, we have

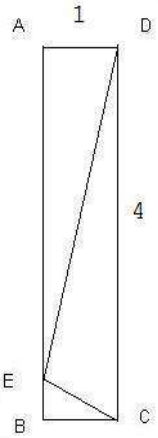
$$\int\int f(x, y) = \int_a^b e^{2i\pi x} dx \int_c^d e^{2i\pi y}$$

$$= \left(\frac{e^{2i\pi b} - e^{2i\pi a}}{2i\pi}\right)\left(\frac{e^{2i\pi d} - e^{2i\pi c}}{2i\pi}\right)$$

Thus, the integral of f over a rectangle is zero if and only if it has at least one integer side is zero; hence, in case of a tiling by such rectangles, the integral is zero which means that the big rectangle has an integer side.

Can the unit square can be tiled by triangles similar to the right-angled triangle with angles $30^\circ, 60^\circ$ and 90°?
The answer turns out to be 'No'!
On the other hand, the unit square can be tiled by triangles similar to the right-angled triangle with angles $15^\circ, 75^\circ$ and 90° (!)

Here is a figure showing how. We have tiled a rectangle of size 1×4 here; we may join 4 such rectangles to get a square!



$AE = 2 + \sqrt{3}$

$BE = 2 - \sqrt{3}$

$AD = BC = 1$

$AB = CD = 4$

$DE = 2\sqrt{(2 + \sqrt{3})}$

$CE = 2\sqrt{(2 - \sqrt{3})}$

angle(AED)= 15 degrees
angle(BCE)= 15 degrees
angle(EDC)= 15 degrees
angle(DEC)= 90 degrees

$\Delta\ DAE \sim \Delta\ EBC \sim \Delta\ CED$

The three triangles DAE, EBC, CED above similar to the 15,75,90 triangle have sides

$$(1, 2+\sqrt{3}, 2\sqrt{2+\sqrt{3}}),$$

$$(2-\sqrt{3}, 1, 2\sqrt{2-\sqrt{3}}); \textit{and}$$

$$(2\sqrt{2+\sqrt{3}}, 2\sqrt{2-\sqrt{3}}, 4)$$

respectively.

A square that can be tiled by triangles similar to a given triangle T is:
either a right triangle such that $\tan(\theta)$ is an algebraic number with all conjugates positive (where θ is a non-right-angle of the triangle)
or a triangle whose angles are $15^\circ, 45^\circ$ and 120°
or $45^\circ, 60^\circ$ and 75°
or $22.5^\circ, 45^\circ$ and 112.5°.

We must mention the idea of using electric networks to solve tiling problems involving rectangles.

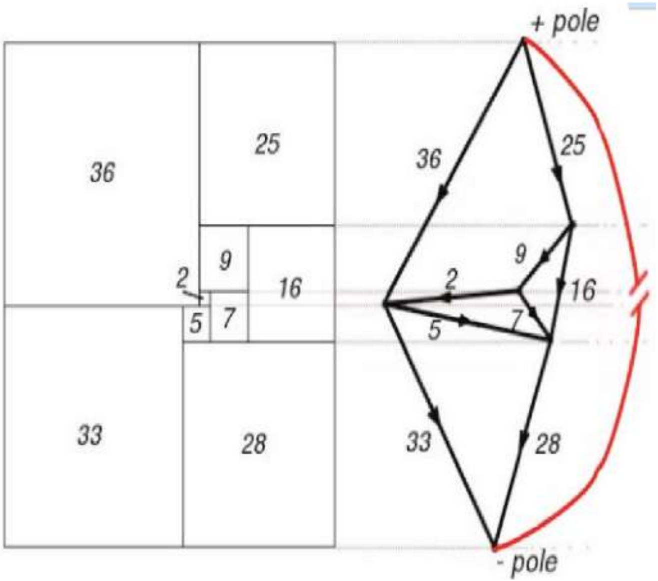
As early as in 1903, Max Dehn showed that a rectangle can be tiled by squares if, and only if, the aspect ratio of the rectangle is a rational number, and this was reproven in 1940 by four mathematician who associated a direct current circuit to each tiling.

The idea is to represent each horizontal line segment in the drawing of the squared rectangles as a dot; each dot represents a terminal in the electrical circuit.

A line connecting two of the terminals is then the square that has those two horizontal lines as boundaries.

Kirchoff's laws (among other laws) describe how the current must behave if the circuit is to be complete.

Kirchoff's first law asserts that the sum of the currents flowing at any of the terminals must be zero, and the second law asserts that the sum of the currents for the entire circuit has to be zero.


36
25
9
2
16
5
7
33
28
+ pole
36
25
9
2
16
7
5
33
28
- pole


99
78
21
57
43
77
16
9
34
25
41

Consider any n-digit integer.
Divide it into a right part of r digits and a left part of n-r digits; too the left part add a number $L < 10$ and to the right part add some $R < 10$.
The addition is done modulo 10 and the "carry-over" is ignored.
Transfer the left part to the right of the right part and we again get an n-digit number.
Apply this same process to the new number.
Iterating this several times, we can ask if we get the original number back, and, if so, what is the least number N of steps required? Here is an example.

Example: We take $n = 8, r = 2, L = 4, R = 2.$
Starting with the number 56240317*, the iteration gives*

56240317
19562407
09195628
etc.

56240317	26051556	07175426
19562407	58260519	28071758
09195628	11582609	50280711
20091950	01115820	13502801
52200913	22011152	03135022
15522003	54220115	24031354
05155224	17542205	56240317

which gives back the original number at the 20-th step; I will leave you to discover for yourself that the minimal number of steps in all cases is $10n/(n,r))(10, L+R)$.

We can prove in any base b that $N = \frac{bn}{(b,L+R)(n,r)}$ where (a,b) denotes the G.C.D. of two numbers a and b.
Let us denote the positions of the n digits from left to right by $1, 2, \cdots, n$, respectively. The positions change as $a \mapsto a + r \mapsto a + 2r \cdots$ for each $a < n$, where $+$ is addition modulo n. For repetition of the original number, we should have some $k > 0$ so that $a + kr \equiv a \bmod n$. In other words, we are looking at the least such k - that is, at the order k of r in the additive group of integers modulo n. Clearly, $k = n/(n,r)$. The choice of k only ensures that the positions of the original digits are the same after every k steps.

Now, for any $m \leq k = n/(n, r)$, there is a corresponding a_0 such that $a_0 + mr = n$. We have

$$a_0 \mapsto a_0 + r \mapsto \cdots a_0 + (m-1)r = n - r$$

$$\overset{L}{\mapsto} a_0 + mr = n \overset{R}{\mapsto} a_0 + (m+1)r \cdots \mapsto a_0 + kr \equiv a_0$$

Thus, we have an increment of $L + R$ in the value of each digit after every k steps. For repetition of the original number, this increment should be a multiple of b and, therefore, N must be a multiple of k as well as of $kb/(L + R)$. This gives the smallest N to be the L.C.M. of k and $kb/(L + R)$.
So, $N = \frac{bn}{(b,L+R)(n,r)}$.

If a monkey were to use a typewriter with, say t characters. What are the chances that it will eventually type a string of letters matching precisely with the full text of Hamlet, say?

If Hamlet has a total of N characters, say, then picking an arbitrary starting point of the typing, the chance that it is the first letter of the Hamlet text is $(1/t)^N$.

If E_i is the event that the i-th character of the starting point of Hamlet, the events E_1, E_{N+1}, E_{2N+1} etc. are independent and a powerful result called Borel-Cantelli lemma assures us that not only will the monkey eventually type the Hamlet, it will do so infinitely many times!

Lattice polygons

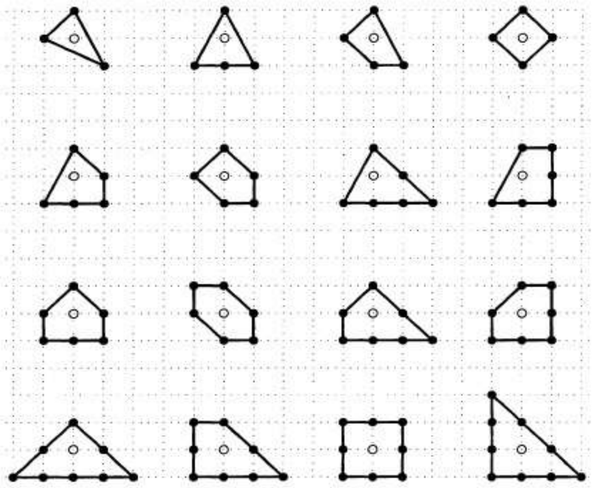
The title will immediately elicit a response like 'yes, I know Pick's theorem.' There is much more to be discovered actually.

If a convex polygon with lattice points as vertices contains $i > 0$ lattice points in the interior and b boundary lattice points, Peter Scott showed very elementarily the remarkable fact that $b \leq 2i + 7$.

This means that if we call legal polygons to be those with exactly one lattice point inside, say $(0, 0)$, the number of legal polygons is severely restricted.

Indeed, Scott determined all of them up to 'modular' transformations; these are the:

Sweet Sixteen



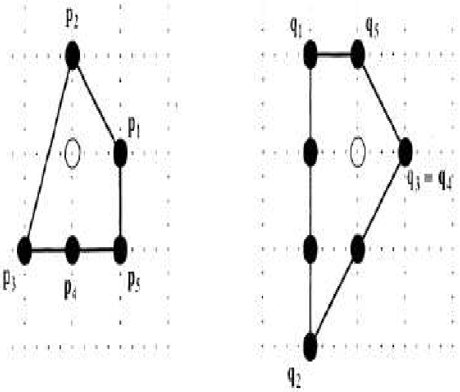
The 16 equivalence classes of legal polygons.

p. 240

A beautiful fact using the above list was observed around the year 2000 and the conceptual explanation requires some mathematics of high level.

If P is any legal polygon, with vertices $P_1, P_2, \cdots, P_n$, one defines the dual polygon $\hat{P}$ to be the smallest convex set containing the points $Q_1 = P_2 - P_1, Q_2 = P_3 - P_2, \cdots, Q_n = P_1 - P_n$ where this difference is thought of like vectors.

For instance, the figure here shows a legal polygon and its dual (note that two of the Q_i's can cincide!):



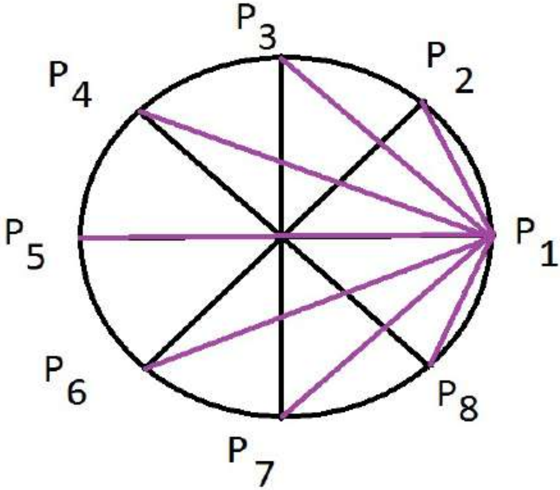
Note the remarkable fact that the ‘discrete lengths’ of the boundary segments - by which we mean one less than the number of lattice points on a boundary segment including the end points - are 5 and 7 adding up to 12.

The reasonable this is remarkable is that this is so for ANY legal polygon and its dual!

The mathematicians who observed this ‘proved it conceptually’ using Ramanujan’s cusp form of weight 12 (hence the 12 here!) among other things.

Start with the following simple question:

On the unit circle, take n points dividing the circumference into n equal parts. From one of these n points, draw the $n-1$ chords joining it to the other points. What is the product of the lengths of these chords?.


P_3
P_2
P_4
P_5
P_1
P_6
P_8
P_7

A more difficult problem is to start from one of the points and - go in one direction (say, the anticlockwise direction) - and draw the chords joining it to the k-th point from it for each k relatively prime to n. What is the product of the lengths of these chords in this case?

The answer turns out to be p or 1 according as to whether n is a power of a prime p or not!

Answer to the chord length problem is as follows.

We may assume that the origin is the centre and that points are $P_{d+1} = e^{2id\pi/n}$ for $d = 0, 1, \cdots, n-1$.

Note that the product of lengths of all the chords $P_1 P_i$ is simply $\prod_{d=1}^{n-1} |1 - e^{2id\pi/n}| = n$ by evaluating $\frac{X^n - 1}{X - 1}$ at $X = 1$. In fact,

$$P(n) := \prod_{l=1}^{n-1} (1 - \zeta^l) = n.$$

Denoting our product $\prod_{(d,n)=1}(1-\zeta^d)$ by $Q(n)$, where $\zeta = e^{2i\pi/n}$, we can see that

$$P(n) = \prod_{r|n} Q(r).$$

By Möbius inversion, $Q(n) = \prod_{d|n} P(d)^{\mu(n/d)} = \prod_{d|n} d^{\mu(n/d)}$.

The function

$$\log Q(n) = \sum_{d|n} \mu(n/d)\log(d)$$

can be identified with the von Mangoldt function $\Lambda(n)$ which is defined to have the value $\log(p)$ if n is a power of p and 0 otherwise.

Using this identification, exponentiation gives us the value $Q(n) = p$ or 1 according as to whether n is a power of a prime p or not.

To see why $\Lambda(n) = \sum_{d|n} \mu(n/d) \log(d)$, we write $n = \prod_{p|n} p^{v_p(n)}$ and note that

$$\log(n) = \sum_{p|n} v_p(n) \log(p)$$

The right hand side is clearly $\sum_{d|n} \Lambda(d)$.

Hence, Möbius inversion yields

$$\Lambda(n) = \sum_{d|n} \log(d)\mu(n/d).$$

This famous problem asks us to prove that the transformation

$$P : (a, b, c, d) \mapsto (|a - b|, |b - c|, |c - d|, |d - a|)$$

on integers, leads any (a, b, c, d) to $(0, 0, 0, 0)$ after finitely many steps.

This turns out to be true for any power of 2 in place of 4. I will talk about a very interesting generalization.

Consider the same P on $\mathbf{R}^4$ now; one can show that it has a fixed point of the form

$$(1, 1+\lambda, (1+\lambda)^2, (1+\lambda)^3)$$

where λ is the unique real root of the cubic $x^3 + 2x^2 - 2$.

(Difficult) The above 4-tuple and its positive scalar multiples are the only ones which do NOT lead to $(0,0,0,0)$.

The proof depends on the subgroup of order 8 in S_4 generated by the two transformations

$$(a, b, c, d) \mapsto (d, c, b, a)$$

and

$$(a, b, c, d) \mapsto (b, c, d, a)$$

The basic reason behind the above result can be easily arrived at: If we arrange a 4-tuple in non-decreasing order (a, b, c, d), then the transformation produces the 4-tuple

$$(b - a, c - b, d - c, d - a)$$

In other words, we have $(r, s, t, r + s + t)$ for some positive numbers.

The analysis leads to looking at 3-tuples and to the 3×3 matrix

$$\begin{matrix} -1 & 1 & 0 \\ 0 & -1 & 1. \\ 1 & 1 & 0 \end{matrix}$$

Its characteristic polynomial is $x^3 + 2x^2 - 2$ whose unique real root is λ above.

Nim

Place buttons or coins in 3 rows (there can be different number of coins in each row).
For example, there could be 9,5 and 12 coins.
Two players play this game.
First, a player takes some coins from any row (she is allowed to take coins only from one row and she has to take at least one coin but she can take any number of coins from that row including all the coins on that row also).
The next player does the same (takes coins from any one row).
This way, they alternate.
Finally, the player who removes the last coin is the winner.

Here also, the strategy is based on writing any number in binary form.
For example, the numbers $9, 5, 12$ are:

$$1001$$

$$0101$$

$$1100$$

I have deliberately written 0101 instead of 101 for 5 so that the strategy can be explained easily.
The sums of the columns from left to right are:
$2, 2, 0, 2$. So, all of them happen to be even.

Let us call this a *safe combination*.
We claim:
The person whose turn it is to play facing a safe combination, can be made to lose.
Moreover, if at some stage the numbers written in binary form have a non-safe combination (that is, at least one column sum is odd), then the player to play now, can play in such a way that she can make this into a safe combination.
For example, if we have $9, 5, 12$ coins and your opponent has to play. you can defeat her.

For instance, if she removes 2 coins from the 9 coins, the rows have $7, 5, 12$ which are:

$$0111$$

$$0101$$

$$1100$$

To transform the existing column sums $1, 3, 1, 2$ to a safe combination, you need to make them $0, 2, 2, 2$; that is,
the last row consisting of the biggest number should change from 1100 to 0010.
So, you should leave out just 2 coins in the last row of 12 (that is, take away 10 coins).
From this new safe combination

$$111$$

$$101$$

$$010$$

Whatever your opponent does, she has to make an unsafe combination because she can remove coins only from one row which means she makes at least 1 on that row to be 0.

To summarize, each time, you just add the columns in binary and find there is a unique binary expansion "for the largest row" which makes the combination safe.
Change your row to the number of coins corresponding to this unique binary expansion.

Wythoff's game

A variant of Nim is when we have two piles of coins and two players alternately make a move within the following rules.

Each move consists of removal of any non-zero number of coins from either one of the two piles or of removal of the same number of coins from both piles.

The player removing the last coins wins.

For instance, after your move, if you leave the two piles with 1 and 2 coins, then your opponent can clearly be made to lose.

Here the strategy is provided by Fibonacci numbers!

It turns out that every natural number has a unique expression (observed by Zeckendorf) as a sum of Fibonacci numbers no two adjacent.

Numbering the Fibonacci numbers as $F_1 = 1, F_2 = 2, F_3 = 3$ etc., the Fibonacci-base expansion of

$$46 = 34 + 8 + 3 + 1 = F_8 + F_5 + F_3 + F_1 = (10010101)_F.$$

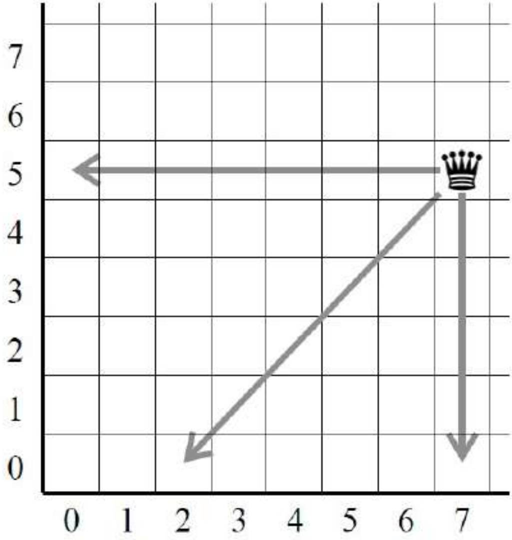
The expression is easy to obtain using the greedy algorithm!

If you leave $a \leq b$ coins in the two piles, then you are in a winning position if, and only if,

The Zeckendorf expression of a ends in an even number of zeroes, and that of b is just the expression of a followed by 0 at the end; example $(a, b) = (11, 18)$.

Also, from such a position, your opponent has to make a move which ends again in a safe position for you where you can force an unsafe position on her!

Wythoff's game can be represented graphically with $(1/4)$-th of an infinite chessboard as in the figure below.



7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7

A queen is placed on some square of the board. Each player in turn moves the queen to some other square, where the requirement is that the queen can move only move left, down, or diagonally southwest. The player who takes the queen to the corner wins.

A puzzle popularized by Sam Lloyd (which was not originally due to him!) asked for a solution to a problem for which he offered a thousand dollars in 1879; it goes as follows.

Look at the picture here of a 4×4 square on which 15 coins have been placed leaving out the last square empty.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

The idea is to slide the coins utilizing the empty square and to find out what kind of arrangements are possible. Sam Lloyd offered 1000 dollars to anyone who could get the arrangement with 14 & 15 switched above. To get a feeling for this the natural thing is to first look at the simplest analogue of it viz. the figure:

1	2
3	

A moment's thought would convince the reader that the pattern

1	3
2	

can never occur. In fact, only those arrangements are possible where $1, 2, 3$ occur in that order (if we go clockwise).

The technology which might allow us to analyze the puzzle systematically is in the form of permutations.

Let us use the convention that $\sigma\tau$ denotes the permutation where σ is applied after τ. For instance, in S_3, look at $\sigma = (1,2)$ $\tau = (2,3)$. Then $\sigma\tau = (1,2,3)$.

Look at the bottom right corner of the Sam Lloyd puzzle

11	12
15	

By sliding cyclically 4 times, we can get to:

15	11
12	

Calling the empty slot as 16, this is the permutation

$$(11, 15, 12) = (12, 16)(15, 16)(11, 16)(12, 16).$$

The puzzle amounts to applying transpositions of the form $(a, 16)$ repeatedly. In the original puzzle, suppose it is possible to write

$$(14, 15) = (a_1, 16) \cdots (a_n, 16)$$

Now, as 16 moves left and right and top and up and down to return to its original position, the right side has even number of transpositions. This is a contradiction. Hence the original puzzle cannot be solved. Now, we show that every even permutation of $1, \cdots, 16$ which fixes 16 can be obtained!
First, it is easy to see that A_n is not only generated by all the 3-cycles but even by only the 3-cycles of the form $(1, 2, i)$ as i varies (where $n \geq 3$).
Indeed,

$$(i, j, k) = (1, 2, k)(1, 2, j)^2(1, 2, i)(1, 2, k)^2.$$

Now, come back to our 4×4 puzzle. We will show that all $(11, 12, i)$ are possible. If we can get a sequence of moves g which moves i to 15 and fixes $11, 12, 16$, then clearly

$$(11, 12, i) = g^{-1}(11, 12, 15)g$$

1	2	3	4
5	6	7	8
9	10		11
13	14	15	12

instead of with the original

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

In fact, $1, 5, 9, 13, 14, 15, 7, 3, 2$ can be moved to 15 through a path involving the 1st and 3rd columns and 1st and 4th rows. One can similarly move $7, 8, 4, 3, 2, 6, 10, 14, 15$.

This proves that all even permutations of $1, \cdots, 15$ are possible.

Some questions that teachers need to test themselves with

Two years back, there was an investigation into the role of attention in the reflective thinking of school mathematics teachers from various countries. It analyzed the teacher's ability to pay attention to detail and use her mathematical knowledge. The vast majority of teachers can be expected to have an excellent knowledge of mathematical techniques. The question examined in that testing was whether this kind of knowledge might structure their attention in such a way that the emphasis on procedures deflects their attention from the essential details. Participant teachers were given a mini-test containing seven simple mathematics questions.

Question 1.
Find the area of the right-angled triangle if its hypotenuse is 10 cm and the height to the hypotenuse is 6 cm long.

Question 2.
Find the domain of the function $y = f(g(x))$ where $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-2}$.

Question 3.
Solve the equation $\log(x^2 + 17x - 18) - \log(x^2 + 5x - 6) = 0$.

Question 4.
Prove the identity $\sin(x) = \sqrt{(1 - \cos^2(x))}$.

Question 5.
Show that $\frac{x^2+\sqrt{x}+1}{x-1} = 0$ has a solution in the interval $[0, 2]$.

Question 6.
Find the derivative of the function $y = \log(2\sin(3x) - 4)$.

Question 7.
Determine the integral $\int_{-1}^{1} \frac{dx}{x}$.

Most questions in the test were provocative in the sense that they looked like routine questions but, in fact, had some catch. The results of the test were startling as the vast majority of the participants gave incorrect answers to most questions in the test!

To the question:
"Find the area of the right-angled triangle if its hypotenuse is 10 cm and the height to the hypotenuse is 6 cm long", the answer is:

There is no meaning to this question because the hypotenuse in a right triangle is the diameter to its semicircle and so, the height can be at the most 5 cm.

The second question was:
Find the domain of the function $y = f(g(x))$ where $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-2}$.

The correct answer $x \geq 2$ is where g and $f \circ g$ are defined. Many teachers simplified $f(g(x))$ to $x - 1$ and got the wrong answer.

Question 3 was:
Solve the equation $\log(x^2 + 17x - 18) - \log(x^2 + 5x - 6) = 0$.

There is no solution (many teachers did not realize that $x = 1$ is outside the domain of both log functions).

The 4th question was:
Prove the identity $\sin(x) = \sqrt{(1 - \cos^2(x))}$.

This is not an identity at all; the solutions are the unions of $[2n\pi, (2n+1)\pi]$.

The next question was:
Show that the equation $\frac{x^2+\sqrt{x}+1}{x-1} = 0$ has a solution in the interval $[0, 2]$.

There are no solutions as $x^2 + \sqrt{x} + 1$ is always positive. Most teachers checked that values of $\frac{x^2+\sqrt{x}+1}{x-1}$ at 0 and at 2 are negative and positive respectively and falsely applied IMVT not realizing that the function is not continuous in $[0, 2]$.

The 6th question is:
Find the derivative of the function $y = \log(2\sin(3x) - 4)$.

Again, the derivative does not exist as the function itself does not exist - the argument of the log is negative! Most teachers applied chain rule without looking at the domain.

Question number 7 was:
Determine the integral $\int_{-1}^{1} \frac{dx}{x}$.

Here also, the Newton-Leibniz formula is not applicable as the function $1/x$ is not continuous in $[-1, 1]$. So, this is not a definite integral.

Finally, if I were able to present in person, I would have shown you some hands-on demonstrations but due to the constraints, I point out two videos by Tadoshi Tokieda which would be very enjoyable to watch and learn from.

https://www.youtube.com/watch?v=CN8hK3YFqhM
https://www.youtube.com/watch?v=pkfDYOZ1p4Y

THANK YOU FOR YOUR PATIENCE!