Prime divisors in an arithmetic progression

Shanta Laishram

Associate Professor of Mathematics
Theoretical Statistics and Mathematics Unit
Indian Statistical Institute, New Delhi, India

E-mail: shanta@isid.ac.in, shantalaishram@gmail.com
Homepage: http://www.isid.ac.in/∼shanta

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Observation

- Well-known: A product of $k \geq 1$ consecutive positive integers is divisible by $k!$.
- One of the combinatorial proofs is given by the fact that the binomial coefficient $\binom{n}{k} \in \mathbb{Z}$.

$$^nC_k = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

- Each prime $p \leq k$ divides a product of $k \geq 1$ consecutive positive integers.
- We can ask if there is a prime $> k$ dividing a product of k consecutive positive integers.

A result of Sylvester-Erdős

- A well-known theorem of Sylvester-Erdős answers this question.
- *A product of k consecutive integers each of which exceeds k is divisible by a prime greater than k.*
- In other words,

$$P((n+1)(n+2)\cdots(n+k)) > k \quad \text{when} \quad n \geq k.$$

- Here $P(m)$ stands for the largest prime divisor of m with the convention $P(1) = 1$.

A result of Sylvester-Erdős

- This is a non trivial result.
- In fact if $n = 0$, then

$$(n+1)(n+2)\cdots(n+k) = k!$$

  and hence the condition $n \geq k$ cannot be avoided.
- A consequence of this result is the *Bertrand's Postulate*: Given $k > 1$, there is a prime p with $k < p < 2k$.
- We can observe it easily by taking $n = k$. There is a prime $> k$ dividing

$$(k+1)(k+2)\cdots(2k-1)2k$$

  and this prime is of the form $k+i$ with $1 \leq i < k$.

A result of Sylvester-Erdős

- Let $p_1 = 2, p_2 = 3, p_3 = 5, \cdots$ be the sequence of primes. So, p_m is the $m-$th prime.
- Also denote by $\pi(m)$ the number of primes $\leq m$. For example, $\pi(100) = 25$ as there are 25 primes upto 100.
- Hence given k, the least prime exceeding k is $p_{\pi(k)+1}$.
- The assertion $P((n+1)\cdots(n+k)) > k$ does not hold if

$$n < p_{\pi(k)+1} - k.$$

Suffices for k prime

Lemma 1.

Suppose that

(1) $$P\left((n+1)\cdots(n+k)\right) > k \quad \text{for} \quad n \geq k.$$

holds for all primes k. Then it holds for all k.

Proof.

Assume that (1) holds for all primes k. Let $k_1 \leq k < k_2$ with k_1, k_2 consecutive primes. Let $n \geq k$. Then $n \geq k_1$ and $(n+1)\cdots(n+k_1)$ has a prime factor $p > k_1$ by our assumption. Further we observe that $p \geq k_2 > k$ since k_1 and k_2 are consecutive primes. Hence p divides $(n+1)\cdots(n+k_1)\cdots(n+k)$ and (1) holds for k. □

Proof of the result of Sylvester-Erdős

- We now prove

$$(2) \qquad P((n+1)\cdots(n+k)) > k \quad \text{for } k \text{ prime and } n \geq k.$$

- Putting $x = n + k$, we see from $n \geq k$ that $x \geq 2k$ and (2) is equivalent to

$$P\left(\binom{x}{k}\right) = P\left(\frac{x(x-1)\cdots(x-k+1)}{k!}\right) > k.$$

- Therefore we will prove

$$(3) \qquad P\left(\binom{x}{k}\right) > k \quad \text{for} \quad k \text{ prime and } x \geq 2k.$$

- We will prove in an elementary way without using prime number theory.

Proof of the result of Sylvester-Erdős

- Suppose for some k prime and $x \geq 2k$,

$$(4) \qquad P\left(\binom{x}{k}\right) = P\left(\frac{x(x-1)\cdots(x-k+1)}{k!}\right) \leq k.$$

Then

$$(5) \qquad P\left(x(x-1)\cdots(x-k+1)\right) \leq k.$$

Power of prime dividing a binomial coefficient

Lemma 2.

Let $p^a \Big| \binom{x}{k}$. Then $p^a \leq x$.

Proof.

We observe that

$$\mathrm{ord}_p \binom{x}{k} = \sum_{l=1}^{\infty} \left(\left[\frac{x}{p^l} \right] - \left[\frac{x-k}{p^l} \right] - \left[\frac{k}{p^l} \right] \right).$$

Each of the summand is at most 1 if $p^l \leq x$ and 0 otherwise. Therefore $\mathrm{ord}_p \binom{x}{k} \leq s$ where $p^s \leq x < p^{s+1}$. Thus

$$p^a \leq p^{\mathrm{ord}_p \binom{x}{k}} \leq p^s \leq x.$$

□

x bounded in terms of k

Lemma 3.

We have

$$P\left(\binom{x}{k}\right) \leq k \quad \text{implies} \quad x < k^{\frac{k}{k-\pi(k)}}. \tag{6}$$

Further

$$x < k^2 \quad \text{for } k \geq 11 \quad \text{and} \quad x < k^{\frac{3}{2}} \quad \text{for } k \geq 37.$$

x bounded in terms of k

Proof.

We have

$$\binom{x}{k} = \frac{x}{k}\frac{x-1}{k-1}\cdots\frac{x-k+1}{1} > \left(\frac{x}{k}\right)^k.$$

From $P\left(\binom{x}{k}\right) \le k$ and $p^a \le x$ for $p^a | \binom{x}{k}$, we get $\binom{x}{k} \le x^{\pi(k)}$. Comparing the bounds for $\binom{x}{k}$, we have

$$\frac{x^k}{k^k} < \binom{x}{k} < x^{\pi(k)} \quad \text{implying} \quad x < k^{\frac{k}{k-\pi(k)}}.$$

For $k \ge 11$, we exclude 1 and 9 to see that there are at most $[\frac{k+1}{2}] - 2$ odd primes upto k. Hence $\pi(k) \le 1 + [\frac{k+1}{2}] - 2 \le \frac{k}{2}$ for $k \ge 11$ giving $x < k^2$ for $k \ge 11$. □

x bounded in terms of k

Proof.

Further the number of composite integers $\leq k$ and divisible by 2 or 3 or 5 is

$$\begin{aligned}
&\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor + \lfloor \frac{k}{5} \rfloor - \lfloor \frac{k}{6} \rfloor - \lfloor \frac{k}{10} \rfloor - \lfloor \frac{k}{15} \rfloor + \lfloor \frac{k}{30} \rfloor - 3 \\
\geq &\frac{k}{2} + \frac{k}{3} + \frac{k}{5} + \frac{k}{30} - \frac{k}{6} - \frac{k}{10} - \frac{k}{15} - 7 \\
= &\frac{11}{15}k - 7.
\end{aligned}$$

Thus we have $\pi(k) \leq k - 1 - (\frac{11}{15}k - 7) \leq \frac{k}{3}$ for $k \geq 90$. By direct computation, we see that $\pi(k) \leq \frac{k}{3}$ for $37 \leq k < 90$. Hence $\frac{k}{k-\pi(k)} \leq \frac{3}{2}$ for $k \geq 37$. □

x bounded in terms of k

Lemma 4.

We have

(7) $$P\left(\binom{x}{k}\right) \leq k \quad \text{implies} \quad x < k^{\frac{k}{k-\pi(k)}}.$$

Further

$$x < k^2 \quad \text{for } k \geq 11 \quad \text{and} \quad x < k^{\frac{3}{2}} \quad \text{for } k \geq 37.$$

Lemma 5.

Let $x < k^{\frac{3}{2}}$. Then

(8) $$P\left(\binom{x}{k}\right) \leq k \quad \text{implies} \quad \binom{x}{k} < 4^{k+\sqrt{x}}.$$

Proof of Lemma 5

- Assume

$$\prod_{p\le k} p \prod_{p\le\sqrt{k}} p \prod_{p\le\sqrt[3]{k}} p \cdots < 4^k. \tag{9}$$

- Taking $k = \sqrt{x}$,

$$\prod_{p\le\sqrt{x}} p \prod_{p\le\sqrt[4]{x}} p \prod_{p\le\sqrt[6]{x}} p \cdots < 4^{\sqrt{x}}.$$

- Since $x < k^{\frac{3}{2}}$, we have $\sqrt[2l-1]{x} \le \sqrt[l]{k}$ for $l \ge 2$ and

$$\binom{x}{k} = \prod_{p^a \| \binom{x}{k}} p^a \le \prod_{p\le k} p \prod_{p\le\sqrt{x}} p \prod_{p\le\sqrt[3]{x}} p \cdots$$

$$\le \left(\prod_{p\le k} p \prod_{p\le\sqrt{k}} p \prod_{p\le\sqrt[3]{k}} p \cdots\right)\left(\prod_{p\le\sqrt{x}} p \prod_{p\le\sqrt[4]{x}} p \prod_{p\le\sqrt[6]{x}} p \cdots\right)$$

$$< 4^{k+\sqrt{x}}.$$

Proof continued: Upper bounds for prime divisors

- We prove

$$(10) \qquad \prod_{p \le k} p \prod_{p \le \sqrt{k}} p \prod_{p \le \sqrt[3]{k}} p \cdots < 4^k.$$

- For every prime p and a positive integer a with $k < p^a \le 2k$, we have

$$\mathrm{ord}_p\left(\binom{2k}{k}\right) = \mathrm{ord}_p\left(\frac{(2k)!}{(k!)^2}\right) \ge \sum_{i=1}^{a}\left\{\lfloor\frac{2k}{p^i}\rfloor - 2\lfloor\frac{k}{p^i}\rfloor\right\} \ge 1$$

since

$$\lfloor\frac{2k}{p^i}\rfloor - 2\lfloor\frac{k}{p^i}\rfloor \ge 0 \quad \text{and} \quad \lfloor\frac{2k}{p^a}\rfloor - \lfloor\frac{k}{p^a}\rfloor = 1.$$

Upper bounds for prime divisors

- Let $\lceil \nu \rceil$ denote the least integer greater than or equal to ν. Let $2^{m-1} \leq k < 2^m$ and we put

$$a_1 = \lceil \frac{k}{2} \rceil, \cdots, a_h = \lceil \frac{k}{2^h} \rceil, \cdots, a_m = \lceil \frac{k}{2^m} \rceil = 1.$$

- Then

$$a_1 > \cdots > a_m \quad \text{and} \quad a_h < \frac{k}{2^h} + 1 = \frac{2k}{2^{h+1}} + 1 \leq 2a_{h+1} + 1$$

implying $a_h \leq 2a_{h+1}$.

- Also $2a_2 < \frac{k}{2} + 2 \leq a_1 + 2$ and therefore $2a_2 \leq a_1 + 1$.

Upper bounds for prime divisors

- Since $2a_1 \geq k$, we see that

$$(1, k] \subseteq \cup_{h=1}^{m} (a_h, 2a_h].$$

- Let p and r be such that $p^r \leq k < p^{r+1}$ and let $1 \leq i \leq r$. Then $p^i \leq k$ and there exists h_i such that

$$a_{h_i} < p^i \leq 2a_{h_i} \quad \text{and} \quad p \Big| \binom{2a_{h_i}}{a_{h_i}}.$$

- Observe: $a_{h_i} \neq a_{h_j}$ for $1 \leq j < i \leq r$ since $pa_{h_j} < p^{j+1} \leq p^i \leq 2a_{h_i}$. Thus

$$p^r \Big| \binom{2a_1}{a_1} \binom{2a_2}{a_2} \cdots \binom{2a_m}{a_m}.$$

- Hence

$$\prod_{p\le k} p \prod_{p\le k^{\frac{1}{2}}} p \cdots = \prod_{p^r\le k<p^{r+1}} p^r \le \binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m}.$$

- Suffices to show:

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m} < 4^k. \tag{11}$$

- Direct calculation: (11) holds for $k \le 10$.
- For example, when $k = 5$, we have $a_1 = 3$, $a_2 = 2$, $a_3 = 1$ so that

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\binom{2a_3}{a_2} = 20 \times 6 \times 2 < 4^5.$$

- Let $k > 10$ and (11) holds for any integer less than k. Then

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m} < \binom{2a_1}{a_1}4^{2a_2-1}. \tag{12}$$

- For $k \geq 8$, we have

$$\binom{2k}{k} < 4^{k-1}.$$

- Hence

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m} < 4^{a_1-1+2a_2-1}$$

and (11) follows since $2a_1 \leq k+1$ and $2a_2 \leq a_1 + 1$. □

Lemma 6.

For $k > 1$, we have

$$\binom{2k}{k} < \frac{4^k}{\sqrt{2k}}. \tag{13}$$

Proof.

$$1 > \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{(2k-2)^2}\right)$$
$$= \frac{1\cdot 3}{2^2}\frac{3\cdot 5}{4^2}\cdots\frac{(2k-3)(2k-1)}{(2k-2)^2}$$
$$> \frac{1}{2k-1}\left(\frac{3\cdot 5\cdots(2k-1)}{2^k k!}\cdot 2k\right)^2 > \frac{4k^2}{2k}\left(\frac{(2k)!}{4^k (k!)^2}\right)^2.$$

□

Lemma 7.

For $k > 1$, we have

(14) $$\binom{2k}{k} > \frac{4^k}{2\sqrt{k}}$$

Proof.

For $k > 1$, we have

$$\begin{aligned} 1 &> \left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\cdots\left(1-\frac{1}{(2k-1)^2}\right) \\ &= \frac{2\cdot 4}{3^2}\frac{4\cdot 6}{5^2}\cdots\frac{(2k-2)2k}{(2k-1)^2} \\ &> \frac{1}{4k}\left(\frac{2^k k!}{3\cdot 5\cdots(2k-1)}\right)^2 = \frac{1}{4k}\left(\frac{4^k (k!)^2}{(2k)!}\right)^2. \end{aligned}$$

x bounded in terms of k

- We have

$$(15) \qquad P\left(\binom{x}{k}\right) \leq k \quad \text{implies} \quad x < k^{\frac{k}{k-\pi(k)}}.$$

  Further

$$x < k^2 \quad \text{for } k \geq 11 \quad \text{and} \quad x < k^{\frac{3}{2}} \quad \text{for } k \geq 37.$$

- Let $x < k^{\frac{3}{2}}$. Then

$$(16) \qquad P\left(\binom{x}{k}\right) \leq k \quad \text{implies} \quad \binom{x}{k} < 4^{k+\sqrt{x}} < 4^{k+k^{\frac{3}{4}}}.$$

- For $k > 1$, we have

$$(17) \qquad \frac{4^k}{2\sqrt{k}} < \binom{2k}{k} < \frac{4^k}{\sqrt{2k}}.$$

x bounded in terms of k

- We have

  (15) $$P\left(\binom{x}{k}\right) \le k \quad \text{implies} \quad x < k^{\frac{k}{k-\pi(k)}}.$$

  Further

  $$x < k^2 \quad \text{for } k \ge 11 \quad \text{and} \quad x < k^{\frac{3}{2}} \quad \text{for } k \ge 37.$$

- Let $x < k^{\frac{3}{2}}$. Then

  (16) $$P\left(\binom{x}{k}\right) \le k \quad \text{implies} \quad \binom{x}{k} < 4^{k+\sqrt{x}} < 4^{k+k^{\frac{3}{4}}}.$$

- For $k > 1$, we have

  (17) $$\frac{4^k}{2\sqrt{k}} < \binom{2k}{k} < \frac{4^k}{\sqrt{2k}}.$$

- $x \geq 4k$: We have $\binom{4k}{k} \leq \binom{x}{k} < 4^{k+\sqrt{x}} < 4^{k+k^{\frac{3}{4}}}$ and

$$4^{k+k^{\frac{3}{4}}} > \binom{4k}{k} = \binom{2k}{k}\frac{4k(4k-1)\cdots(3k+1)}{2k(2k-1)\cdots(k+1)}$$
$$> \frac{4^k}{2\sqrt{k}}2^k = \frac{8^k}{2\sqrt{k}}$$

- $\frac{5}{2}k < x < 4k$: Here $\binom{\lceil\frac{5}{2}k\rceil}{k} \leq \binom{x}{k} < 4^{k+\sqrt{x}} < 4^{k+2\sqrt{k}}$ and

$$4^{k+2\sqrt{k}} > \binom{2k}{k}\frac{\lceil\frac{5}{2}k\rceil(\lceil\frac{5}{2}k\rceil-1)\cdots(\lceil\frac{5}{2}k\rceil-k+1)}{2k(2k-1)\cdots(k+1)} > \frac{4^k}{2\sqrt{k}}\left(\frac{5}{4}\right)^k$$

- $2k \leq x \leq \frac{5}{2}k$: Here $\binom{2k}{k} \leq \binom{x}{k} < 4^{k+\sqrt{x}} \leq 4^{k+\sqrt{5k/2}}$ and

$$4^{k+\sqrt{5k/2}} > \binom{2k}{k} > \frac{4^k}{2\sqrt{k}}.$$

Final Bounds

- Comparing the bounds, we obtain for $k \geq 11$ and $x < k^{\frac{3}{2}}$ that $x < 4k$.
- Further $k < 103$ for $\frac{5}{2}k < x < 4k$ and $k < 113$ for $2k \leq x \leq \frac{5}{2}k$.
- Recall $x < k^{\frac{k}{k-\pi(k)}}$ and $x < k^2$ for $k \geq 11$ and $x < k^{\frac{3}{2}}$ for $k \geq 37$.
- Combining all those: $k \leq 113$;

$$x < k^2 \text{ for } 11 \leq k \leq 31; \;\; x < 4k \text{ for } 37 \leq k \leq 113$$

and

$$x < k^{\frac{k}{k-\pi(k)}} \quad \text{for } k < 11.$$

Idea of Sylvester-Erdős

- Let $S = \{x, x-1, \cdots, x-k+1\}$.
- For each prime $p \le k$, let $0 \le i_p < k$ be such that $\mathrm{ord}_p(x - i_p)$ is maximal.
- Then for $i \ne i_p$, we have

$$\mathrm{ord}_p(x-i) \le \mathrm{ord}_p(x-i-(x-i_p)) = \mathrm{ord}_p(i-i_p).$$

- Put $S_0 = \{x, x-1, \cdots, x-k+1\} \setminus \{x - i_p : p \le k\}$.
- Then $|S_0| \ge k - \pi(k)$ and

$$\mathrm{ord}_p\left(\prod_{x-i\in S_0}(x-i)\right) \le \mathrm{ord}_p\left(i_p!\,(k-1-i_p)!\right) \le \mathrm{ord}_p((k-1)!)$$

- Thus

$$\left(\prod_{x-i\in S_0}(x-i)\right) \le \prod_{p<k} p^{\mathrm{ord}_p((k-1)!)} \le (k-1)!.$$

- Hence from $|S_0| \ge k-\pi(k)$, we get

$$(x-k+1)^{k-\pi(k)} \le (k-1)! \quad \text{or} \quad x \le k-1+((k-1)!)^{\frac{1}{k-\pi(k)}}.$$

  with the equality only when $k=2$.
- We get $x \le 2, 4, 9, 13$ according as $k = 2, 3, 5, 7$, respectively. This contradicts $x \ge 2k$. Thus $k \ge 11$.
- For $11 \le k \le 31$, we get $x \le 71$.

Finally for $11 \leq k \leq 113$ and

$$x \leq 71 \text{ for } 11 \leq k \leq 31; \; x < 4k \text{ for } 37 \leq k \leq 113$$

we use the following result.

Lemma 8.

Let X be a positive real number and k_0 be a positive integer. Suppose that $p_{i+1} - p_i < k_0$ for any two consecutive primes $p_i < p_{i+1} \leq p_{\pi(X)+1}$. Then

$$P(x(x-1)\cdots(x-k+1)) > k$$

for $2k \leq x \leq X$ and $k \geq k_0$.

Proof.

Let $2k \leq x \leq X$. We may assume that none of $x, x-1, \cdots, x-k+1$ is a prime, since otherwise the result follows. Thus

$$p_{\pi(x)} < x-k+1 < x < p_{\pi(x)+1} \leq p_{\pi(X)+1}.$$

Hence by our assumption, we have

$$k-1 = x-(x-k+1) < p_{\pi(x)+1} - p_{\pi(x)} < k_0,$$

which implies $k-1 < k_0 - 1$, a contradiction. □

- Computation gives:

$$p_{i+1} - p_i < \begin{cases} 21 \text{ for } p_{i+1} \leq 457 = p_{\pi(452)+1} \\ 8 \text{ for } p_{i+1} \leq 79 = p_{\pi(73)+1}. \end{cases}$$

- Apply Lemma 8 as follows:
- For $23 \leq k \leq 31$, take $X = 452, k_0 = 21$;
- For $11 \leq k \leq 31$, take $X = 73, k_0 = 10$.
- This completes the proof.

- Recall Sylvester's result:

$$P((n+1)\cdots(n+k)) > k \quad \text{for } n \geq k.$$

- Moser: $P((n+1)\cdots(n+k)) > \frac{11}{10}k$
- Hanson: $P((n+1)\cdots(n+k)) > 1.5k$ unless $(n,k) = (3,2),(8,2),(6,5)$.
- Faulkner: $P((n+1)\cdots(n+k)) > 2k$ if $n \geq p_{\pi(2k)+1}$.
- For $k = 2$, easy to see: $P((n+1)(n+2)) > 4$ except when $n = 1,2,7$.

- **Laishram and Shorey:** We have

$$P((n+1)\cdots(n+k)) > 1.95k \quad \text{for } n \geq k.$$

unless (n, k) belongs to an explicitly given finite set.

- Here 1.95 cannot be replaced by 2 as $P((k+1)\cdots(k+k)) < 2k$.
- **Laishram and Shorey:** We have

$$P((n+1)\cdots(n+k)) > 1.8k \quad \text{for } n \geq k.$$

unless $n = k \in \{3, 4, 5, 8, 11, 13, 14, 18, 63\}$ or

$$(n, k) \in \{(7, 3), (5, 4), (6, 4), (14, 13), (15, 13)\}.$$

Generalization of Sylvester's Theorem

- **Laishram and Shorey:** We have

$$P((n+1)\cdots(n+k)) > 1.95k \quad \text{for } n \geq k.$$

unless (n, k) belongs to an explicitly given finite set.

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$$(n, k) \in \{(7, 3), (5, 4), (6, 4), (14, 13), (15, 13)\}.$$

Generalization of Sylvester's Theorem

- **Laishram and Shorey:** We also have

$$P((n+1)\cdots(n+k)) > 2k \quad \text{for } n \geq \max\left(k+13, \frac{279}{262}k\right).$$

and

$$P((n+1)\cdots(n+k)) > 1.97k \text{ for } n \geq k+13.$$

- Observe: 1.97 cannot be replaced by 2 since there are arbitrary long chains of consecutive composite positive integers.
- The assumption $n \geq \frac{279}{262}k$ is necessary since $P((n+1)\cdots(n+k)) \leq 2 \times 262$ when $n = 278$ and $k = 262$.

Generalization for arithmetic progression

- Let n, d be positive coprime integers with $d > 1$ and put

$$\Delta(n,d,k) = n(n+d)(n+2d)\cdots(n+(k-1)d).$$

- Observe; $P(\Delta(n,d,2)) = 2$ if and only if $n = 1, d = 2^r - 1$ with $r > 1$.
- Sylvester: $\Delta(n,d,k) > k$ if $n \geq d+k$.
- Let $d = 2$. Then

$$\Delta(n,2,k) = n(n+2)(n+2d)\cdots(n+2(k-1)).$$

- Let $n \leq k$. Then $P(\Delta(n,2,k)) < 2k$ implies $P((n+k)\cdots(n+2k-1)) \leq 2k$.
- This gives $n+k-1 < \max(k+13, \frac{279}{262}k)$ or $n < \max(14, \frac{17}{262}k+1)$.
- We consider $k > 2$ and $d > 2$.

Generalization for arithmetic progression

- Langevin: $P(\Delta(n,d,k)) > k$ if $n > k$.
- Shorey and Tijdeman: $P(\Delta(n,d,k)) > k$ unless

$$(n,d,k) = (2,7,3) \text{ where } P(2 \cdot 9 \cdot 16) = 3.$$

- **Laishram and Shorey:** For $k > 2$ and $d > 2$, we have

$$P(n(n+d)\cdots(n+(k-1)d)) > 2k$$

unless (n,d,k) is given by

$$\begin{aligned} k = 3,\ & n = 1, d = 4, 7; \\ & n = 2, d = 3, 7, 23, 79; \\ & n = 3, d = 61;\ n = 4, d = 23; \\ & n = 5, d = 11;\ n = 18, d = 7; \\ k = 4,\ & n = 1, d = 3, 13;\ n = 3, d = 11; \\ k = 10,\ & n = 1, d = 3. \end{aligned}$$

- This is best known result.

Generalization for arithmetic progression

- **Laishram and Shorey:** $P(\Delta(n,3,k)) > 3k$ when $n > 3k$ and $(n,k) \neq (125,2)$.
- **Laishram and Shorey:** $P(\Delta(n,4,k)) > 4k$ when $n > 4k$ and $n \neq 21,45$ when $k = 2$.
- Conjecture: Fix d. For $n > dk$ and $k > 1$, we have

$$P(n(n+d)\cdots(n+(k-1)d)) > dk$$

  except for finitely many n and k.
- This is open.

Generalization for arithmetic progression

- Though we have an elementary proof of Sylvester's result for consecutive integers, the proof of other results need advanced mathematical ideas from Prime number theory, linear forms in logarithms, Thue equations etc.
- However many of the ideas used in the elementary proof can be generalised and are used in subsequent proofs.

Number of prime divisors

- Let $\omega(m)$ denote the number of distinct prime divisors of m. For example, $\omega(10) = 2$.
- The result of Sylvester is in fact

$$\omega(n(n+1)\cdots(n+k-1)) > \pi(k) \quad \text{if } n > k.$$

- $\omega(n(n+1)) \geq 3$ except when $n = 2^p - 1$, a *Mersenne Prime* or $n = 2^{2^r} + 1$, a *Fermat Prime* or $n \in \{1, 2, 3, 8\}$.
- In fact the famous Catalan Conjecture, now a theorem, states that $x^r = 9 = 3^2, y^s = 8 = 2^3$ are the only solutions of

$$x^r - y^s = 1, \quad \text{with } x > 1, y > 1, r > 1, s > 1.$$

Number of prime divisors

- Let $k \geq 3$. Observe that
$$\omega(n(n+1)\cdots(n+k-1)) = \pi(2k) \quad \text{when } n = k+1.$$
- If $k+1$ is prime and $2k+1$ is composite, then we observe that
$$\omega((k+2)\cdots(2k+1)) = \pi(2k) - 1.$$
- Let $k+1$ be a prime of the form $3r+2$. Then $2k+1 = 3(2r+1)$ is composite. Since there are infinitely many primes of the form $3r+2$, we see that there are infinitely many k for which $k+1$ is prime and $2k+1$ is composite.
- Therefore $\omega(\Delta(n,k)) \geq \pi(2k) - 1$ is valid for infinitely many k. Thus an inequality sharper than $\omega(\Delta(n,k)) \geq \pi(2k) - 1$ for $n > k$ is not valid.

Number of prime divisors

- Saradha and Shorey:

$$\omega(n(n+1)\cdots(n+k-1)) \geq \pi(k) + \lfloor \frac{1}{3}\pi(k) \rfloor + 2 \quad \text{if } n > k$$

unless (n, k) belongs to a finite set.

- **Laishram and Shorey**: For $n > k$, we have

$$\omega(n(n+1)\cdots(n+k-1)) \geq \pi(k) + \lfloor \frac{3}{4}\pi(k) \rfloor - 1 + \delta(k)$$

unless (n, k) belongs to a finite set and

$$\delta(k) = \begin{cases} 2 & \text{if } 3 \leq k \leq 6 \\ 1 & \text{if } 7 \leq k \leq 16 \\ 0 & \text{otherwise.} \end{cases}$$

Number of prime divisors

- **Laishram and Shorey**: Let $n > k$. Then

$$\omega(n(n+1)\cdots(n+k-1)) \geq \min\left(\pi(k) + \lfloor\frac{3}{4}\pi(k)\rfloor - 1 + \delta(k),\ \pi(2k) - 1\right).$$

- **Laishram and Shorey**: Let $(n,k) \neq (6,4)$. Then

$$\omega(n(n+1)\cdots(n+k-1)) \geq \pi(2k) \quad \text{for } n > \frac{17}{12}k.$$

- For $n > k$, we cannot have $\pi(2k)$ or $2\pi(k)$ as a lower bound.

Number of prime divisors

- **Laishram and Shorey**: Let $\epsilon > 0$ and $n > k$. Then there exists a computable number k_0 depending only on ϵ such that for $k \geq k_0$, we have

$$\omega(n(n+1)\cdots(n+k-1)) \geq (2-\epsilon)\pi(k).$$

Grimm's Conjecture

- Suppose $n+1, \cdots, n+k$ are all composite numbers, then there are distinct primes P_j such that $P_j | (n+j)$ for $1 \leq j \leq k$.
- This is open and considered quite difficult.
- This implies: If $n, n+1, \cdots, n+k-1$ are all composite, then $\omega(n(n+1)\cdots(n+k-1)) \geq k$ which is also open.
- Grimm's Conjecture has been verified for all $n \leq 1.9 \times 10^{10}$ and for all k.

- **Schinzel's Hypothesis H**:

  Let $f_1(x), \cdots, f_r(x)$ be irreducible non constant polynomials with integer coefficients and the leading coefficients positive. Assume that for every prime p, there is an integer a such that the product $f_1(a)\cdots f_r(a)$ is not divisible by p. Then there are infinitely many positive integers m such that $f_1(m), \cdots, f_r(m)$ are all primes.

- This is open. Considered quite difficult.
- Schinzel's Hypothesis H implies that $\omega(\Delta(n,d,k)) = \pi(k), \pi(k) = 1$ for $k = 3$ and $k \in \{4,5\}$, respectively for infinitely many pairs (n,d).

Number of prime divisors in an AP

- Shorey and Tijdeman: $\omega(\Delta(n,d,k)) \geq \pi(k)$
- Moree: $\omega(\Delta(n,d,k)) > \pi(k)$ for $k \geq 4$ and $(n,d,k) \neq (1,2,5)$.
- Saradha, Shorey and Tijdeman: For $k \geq 6$,

$$\omega(\Delta(n,d,k)) > \frac{6}{5}\pi(k) + 1$$

unless (n,d,k) belongs to

$$\{(1,2,6),(1,3,6),(1,2,7),(1,3,7),(1,4,7),\\ (2,3,7),(2,5,7),(3,2,7),(1,2,8),(1,2,11),\\ (1,3,11),(1,2,13),(3,2,13),(1,2,14)\}.$$

Number of prime divisors in an AP

- **Laishram and Shorey**: For $k \geq 9$

$$\omega(\Delta(n,d,k)) \geq \pi(2k) - \rho$$

unless n, d, k is given by

$$\begin{cases} n=1,\ d=3,\ k=9,10,11,12,19,22,24,31; \\ n=2,\ d=3,\ k=12;\ n=4,\ d=3,\ k=9,10; \\ n=2,\ d=5,\ k=9,10;\ n=1,\ d=7,\ k=10. \end{cases}$$

where

$$\rho = \rho(d) = \begin{cases} 1 \text{ if } d=2, n \leq k \\ 0 \text{ otherwise.} \end{cases} .$$

Number of prime divisors in an AP

- **Laishram and Shorey**: Let $k \geq 4$. Then

$$\omega(\Delta(n, d, k)) \geq \pi(2k) - 1$$

except at $(n, d, k) = (1, 3, 10)$.
- This proves a conjecture of *Moree*.

Applications

- Binary Recurrence Sequences
- Exponential Diophantine Equations
- A weaker Conjecture: Fix d. For $n > dk$ and $k > 1$, we have

$$P(n(n+d)\cdots(n+(k-1)d)) > \frac{dk}{200}.$$

- This conjecture implies a conjecture of Erdos which states that the equation

$$n(n+d)\cdots(n+(k-1)d) = y^2$$

has no solution in positive integers n, d, k with $\gcd(n, d) = 1$ and $k \geq 2$.

Applications

- Irreducibility of Polynomials
- For example, the *Truncated Exponential Polynomials*

$$E_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

are irreducible for all $n \geq 1$.

Thank you