Prime divisors in an arithmetic progression

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Observation

- Well-known: A product of $k \geq 1$ consecutive positive integers is divisible by $k!$.
- One of the combinatorial proofs is given by the fact that the binomial coefficient $\binom{n}{k} \in \mathbb{Z}$.

\[
nC_k = \binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!}.
\]

- Each prime $p \leq k$ divides a product of $k \geq 1$ consecutive positive integers.
- We can ask if there is a prime $> k$ dividing a product of $k$ consecutive positive integers.
A result of Sylvester-Erdős

• A well-known theorem of Sylvester-Erdős answers this question.

• A product of \( k \) consecutive integers each of which exceeds \( k \) is divisible by a prime greater than \( k \).

• In other words,

\[
P((n + 1)(n + 2) \cdots (n + k)) > k \quad \text{when} \quad n \geq k.
\]

• Here \( P(m) \) stands for the largest prime divisor of \( m \) with the convention \( P(1) = 1 \).
A result of Sylvester-Erdős

- This is a non trivial result.
- In fact if \( n = 0 \), then

\[(n + 1)(n + 2) \cdots (n + k) = k!\]

and hence the condition \( n \geq k \) cannot be avoided.

- A consequence of this result is the Bertrand’s Postulate: Given \( k > 1 \), there is a prime \( p \) with \( k < p < 2k \).

- We can observe it easily by taking \( n = k \). There is a prime \( > k \) dividing

\[(k + 1)(k + 2) \cdots (2k - 1)2k\]

and this prime is of the form \( k + i \) with \( 1 \leq i < k \).
Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ be the sequence of primes. So, $p_m$ is the $m$–th prime.

Also denote by $\pi(m)$ the number of primes $\leq m$. For example, $\pi(100) = 25$ as there are 25 primes upto 100.

Hence given $k$, the least prime exceeding $k$ is $p_{\pi(k)+1}$.

The assertion $P((n+1) \cdots (n+k)) > k$ does not hold if

$$n < p_{\pi(k)+1} - k.$$
Lemma 1.

Suppose that

\[(1) \quad P((n+1) \cdots (n+k)) > k \quad \text{for} \quad n \geq k.\]

holds for all primes \(k\). Then it holds for all \(k\).

Proof.

Assume that (1) holds for all primes \(k\). Let \(k_1 \leq k < k_2\) with \(k_1, k_2\) consecutive primes. Let \(n \geq k\). Then \(n \geq k_1\) and \((n+1) \cdots (n+k_1)\) has a prime factor \(p > k_1\) by our assumption. Further we observe that \(p \geq k_2 > k\) since \(k_1\) and \(k_2\) are consecutive primes. Hence \(p\) divides \((n+1) \cdots (n+k_1) \cdots (n+k)\) and (1) holds for \(k\).
Proof of the result of Sylvester-Erdős

We now prove

\[(2) \quad P\left(\binom{n+1}{k}\cdots\binom{n+k}{k}\right) > k \quad \text{for } k \text{ prime and } n \geq k.\]

Putting \(x = n + k\), we see from \(n \geq k\) that \(x \geq 2k\) and (2) is equivalent to

\[P\left(\binom{x}{k}\right) = P\left(\frac{x(x-1)\cdots(x-k+1)}{k!}\right) > k.\]

Therefore we will prove

\[(3) \quad P\left(\binom{x}{k}\right) > k \quad \text{for } k \text{ prime and } x \geq 2k.\]

We will prove in an elementary way without using prime number theory.
Suppose for some \( k \) prime and \( x \geq 2k \),

\[
P \left( \binom{x}{k} \right) = P \left( \frac{x(x-1) \cdots (x-k+1)}{k!} \right) \leq k.
\]

Then

\[
P (x(x-1) \cdots (x-k+1)) \leq k.
\]
Lemma 2.

Let $p^a \mid \binom{x}{k}$. Then $p^a \leq x$.

Proof.

We observe that

$$\text{ord}_p \binom{x}{k} = \sum_{l=1}^{\infty} \left( \left\lfloor \frac{x}{p^l} \right\rfloor - \left\lfloor \frac{x-k}{p^l} \right\rfloor - \left\lfloor \frac{k}{p^l} \right\rfloor \right).$$

Each of the summand is at most 1 if $p^l \leq x$ and 0 otherwise. Therefore $\text{ord}_p \binom{x}{k} \leq s$ where $p^s \leq x < p^{s+1}$. Thus

$$p^a \leq p^{\text{ord}_p \binom{x}{k}} \leq p^s \leq x.$$
Lemma 3.

We have

\[ P \left( \binom{x}{k} \right) \leq k \quad \text{implies} \quad x < k^{\frac{k}{k - \pi(k)}}. \]  

Further

\[ x < k^2 \quad \text{for } k \geq 11 \quad \text{and} \quad x < k^{\frac{3}{2}} \quad \text{for } k \geq 37. \]
Proof.

We have

\[
\binom{x}{k} = \frac{x(x-1)}{k(k-1)} \cdot \frac{x-k+1}{1} > \left( \frac{x}{k} \right)^k.
\]

From \( P ( \binom{x}{k} ) \leq k \) and \( p^a \leq x \) for \( p^a|\binom{x}{k} \), we get \( \binom{x}{k} \leq x^{\pi(k)} \).

Comparing the bounds for \( \binom{x}{k} \), we have

\[
\frac{x^k}{k^k} < \binom{x}{k} < x^{\pi(k)} \quad \text{implying} \quad x < k^{k-\pi(k)}.
\]

For \( k \geq 11 \), we exclude 1 and 9 to see that there are at most \( \left\lceil \frac{k+1}{2} \right\rceil - 2 \) odd primes upto \( k \). Hence \( \pi(k) \leq 1 + \left\lceil \frac{k+1}{2} \right\rceil - 2 \leq \frac{k}{2} \) for \( k \geq 11 \) giving \( x < k^2 \) for \( k \geq 11 \).
Further the number of composite integers \( \leq k \) and divisible by 2 or 3 or 5 is

\[
\left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k}{5} \right\rfloor - \left\lfloor \frac{k}{6} \right\rfloor - \left\lfloor \frac{k}{10} \right\rfloor - \left\lfloor \frac{k}{15} \right\rfloor + \left\lfloor \frac{k}{30} \right\rfloor - 3
\]

\[
\geq \frac{k}{2} + \frac{k}{3} + \frac{k}{5} + \frac{k}{30} - \frac{k}{6} - \frac{k}{10} - \frac{k}{15} - 7
\]

\[
= \frac{11}{15}k - 7.
\]

Thus we have \( \pi(k) \leq k - 1 - \left( \frac{11}{15}k - 7 \right) \leq \frac{k}{3} \) for \( k \geq 90 \). By direct computation, we see that \( \pi(k) \leq \frac{k}{3} \) for \( 37 \leq k < 90 \). Hence \( \frac{k}{k - \pi(k)} \leq \frac{3}{2} \) for \( k \geq 37 \).
Lemma 4.

We have

\[ P \left( \left( \frac{x}{k} \right) \right) \leq k \quad \text{implies} \quad x < k^{k - \pi(k)}. \]

Further

\[ x < k^2 \quad \text{for} \quad k \geq 11 \quad \text{and} \quad x < k^{3/2} \quad \text{for} \quad k \geq 37. \]
Lemma 5.

Let $x < k^{3/2}$. Then

$$P\left(\binom{x}{k}\right) \leq k \quad \text{implies} \quad \binom{x}{k} < 4^{k + \sqrt{x}}.$$ (8)
Proof of Lemma 5

Assume
\[
\prod_{p \leq k} p \prod_{p \leq \sqrt{k}} p \prod_{p \leq \frac{3}{\sqrt{k}}} p \cdots < 4^k.
\]

Taking \(k = \sqrt{x}\),
\[
\prod_{p \leq \sqrt{x}} p \prod_{p \leq \frac{4}{\sqrt{x}}} p \prod_{p \leq \frac{6}{\sqrt{x}}} p \cdots < 4^{\sqrt{x}}.
\]

Since \(x < k^{\frac{3}{2}}\), we have \(\frac{2^{l-1}}{\sqrt{x}} \leq \sqrt{k}\) for \(l \geq 2\) and
\[
\binom{x}{k} = \prod_{p^a \mid \binom{x}{k}} p^a \leq \prod_{p \leq k} p \prod_{p \leq \sqrt{x}} p \prod_{p \leq \frac{3}{\sqrt{x}}} p \cdots
\]
\[
\leq \left( \prod_{p \leq k} p \prod_{p \leq \sqrt{k}} p \prod_{p \leq \frac{3}{\sqrt{k}}} p \cdots \right) \left( \prod_{p \leq \sqrt{x}} p \prod_{p \leq \frac{4}{\sqrt{x}}} p \prod_{p \leq \frac{6}{\sqrt{x}}} p \cdots \right)
\]
\[
< 4^{k + \sqrt{x}}.
\]
Proof continued: Upper bounds for prime divisors

- We prove

\[
\prod_{p \leq k} p \prod_{p \leq \sqrt{k}} p \prod_{p \leq 3/\sqrt{k}} p \cdots < 4^k.
\] (10)

- For every prime \( p \) and a positive integer \( a \) with \( k < p^a \leq 2k \), we have

\[
\text{ord}_p \left( \binom{2k}{k} \right) = \text{ord}_p \left( \frac{(2k)!}{(k!)^2} \right) \geq \sum_{i=1}^{a} \left\{ \left\lfloor \frac{2k}{p^i} \right\rfloor - 2 \left\lfloor \frac{k}{p^i} \right\rfloor \right\} \geq 1
\]

since

\[
\left\lfloor \frac{2k}{p^i} \right\rfloor - 2 \left\lfloor \frac{k}{p^i} \right\rfloor \geq 0 \quad \text{and} \quad \left\lfloor \frac{2k}{p^a} \right\rfloor - \left\lfloor \frac{k}{p^a} \right\rfloor = 1.
\]
Let \( [\nu] \) denote the least integer greater than or equal to \( \nu \). Let \( 2^{m-1} \leq k < 2^m \) and we put

\[
a_1 = \left\lceil \frac{k}{2} \right\rceil, \ldots, a_h = \left\lceil \frac{k}{2^h} \right\rceil, \ldots, a_m = \left\lceil \frac{k}{2^m} \right\rceil = 1.
\]

Then

\[
a_1 > \cdots > a_m \quad \text{and} \quad a_h < \frac{k}{2^h} + 1 = \frac{2k}{2^{h+1}} + 1 \leq 2a_{h+1} + 1
\]

implying \( a_h \leq 2a_{h+1} \).

Also \( 2a_2 < \frac{k}{2} + 2 \leq a_1 + 2 \) and therefore \( 2a_2 \leq a_1 + 1 \).
Since $2a_1 \geq k$, we see that

$$(1, k] \subseteq \bigcup_{h=1}^{m} (a_h, 2a_h].$$

Let $p$ and $r$ be such that $p^r \leq k < p^{r+1}$ and let $1 \leq i \leq r$. Then $p^i \leq k$ and there exists $h_i$ such that

$$a_{h_i} < p^i \leq 2a_{h_i} \quad \text{and} \quad p \divides \binom{2a_{h_i}}{a_{h_i}}.$$

Observe: $a_{h_i} \neq a_{h_j}$ for $1 \leq j < i \leq r$ since $pa_{h_j} < p^{i+1} \leq p^i \leq 2a_{h_i}$. Thus

$$p^r \divides \binom{2a_1}{a_1} \binom{2a_2}{a_2} \cdots \binom{2a_m}{a_m}.$$
Hence
\[
\prod_{p \leq k} p \prod_{p \leq k^{1/2}} p \cdots = \prod_{p^r \leq k < p^{r+1}} p^r \leq \left( \frac{2a_1}{a_1} \right) \left( \frac{2a_2}{a_2} \right) \cdots \left( \frac{2a_m}{a_m} \right).
\]

Suffices to show:
\[
(11) \quad \left( \frac{2a_1}{a_1} \right) \left( \frac{2a_2}{a_2} \right) \cdots \left( \frac{2a_m}{a_m} \right) < 4^k.
\]

Direct calculation: (11) holds for \( k \leq 10 \).

For example, when \( k = 5 \), we have \( a_1 = 3 \), \( a_2 = 2 \), \( a_3 = 1 \) so that
\[
\left( \frac{2a_1}{a_1} \right) \left( \frac{2a_2}{a_2} \right) \left( \frac{2a_3}{a_2} \right) = 20 \times 6 \times 2 < 4^5.
\]
Let $k > 10$ and (11) holds for any integer less than $k$. Then

\[
\binom{2a_1}{a_1} \binom{2a_2}{a_2} \ldots \binom{2a_m}{a_m} < \binom{2a_1}{a_1} 4^{2a_2-1}.
\]

For $k \geq 8$, we have

\[
\binom{2k}{k} < 4^{k-1}.
\]

Hence

\[
\binom{2a_1}{a_1} \binom{2a_2}{a_2} \ldots \binom{2a_m}{a_m} < 4^{a_1-1+2a_2-1}
\]

and (11) follows since $2a_1 \leq k + 1$ and $2a_2 \leq a_1 + 1$. 

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Prime divisors in an arithmetic progression
Lemma 6.

For $k > 1$, we have

\[
\binom{2k}{k} < \frac{4^k}{\sqrt{2k}}.
\]

(13)

Proof.

\[
1 > \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{(2k - 2)^2}\right) = \frac{1 \cdot 3 \cdot 5}{2^2} \cdots \frac{(2k - 3)(2k - 1)}{(2k - 2)^2}
\]

\[
> \frac{1}{2k - 1} \left(\frac{3 \cdot 5 \cdots (2k - 1)}{2^k k!} \cdot 2k\right)^2 > \frac{4k^2}{2k} \left(\frac{(2k)!}{4^k (k!)^2}\right)^2.
\]
Lemma 7. For \( k > 1 \), we have

\[
\binom{2k}{k} > \frac{4^k}{2\sqrt{k}}
\]

Proof. For \( k > 1 \), we have

\[
1 > \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots \left(1 - \frac{1}{(2k-1)^2}\right)
\]

\[
= \frac{2 \cdot 4 \cdot 6 \cdots (2k - 2)2k}{3^2 \cdot 5^2 \cdots (2k - 1)^2}
\]

\[
> \frac{1}{4k} \left(\frac{2^kk!}{3 \cdot 5 \cdots (2k - 1)}\right)^2 = \frac{1}{4k} \left(\frac{4^k(k!)^2}{(2k)!}\right)^2.
\]
We have

\[(15) \quad P\left(\binom{x}{k}\right) \leq k \quad \text{implies} \quad x < k^{\frac{k}{k-\pi(k)}}.\]

Further

\[x < k^2 \quad \text{for} \quad k \geq 11 \quad \text{and} \quad x < k^{\frac{3}{2}} \quad \text{for} \quad k \geq 37.\]

Let \(x < k^{\frac{3}{2}}\). Then

\[(16) \quad P\left(\binom{x}{k}\right) \leq k \quad \text{implies} \quad \binom{x}{k} < 4^k + \sqrt{x} < 4^k + k^2.\]

For \(k > 1\), we have

\[(17) \quad \frac{4^k}{2\sqrt{k}} < \binom{2k}{k} < \frac{4^k}{\sqrt{2k}}.\]
We have

$$P \left( \binom{x}{k} \right) \leq k \quad \text{implies} \quad x < k^{\frac{k}{k - \pi(k)}}. \quad (15)$$

Further

$$x < k^2 \quad \text{for} \ k \geq 11 \quad \text{and} \quad x < k^{\frac{3}{2}} \quad \text{for} \ k \geq 37.$$

Let $x < k^{\frac{3}{2}}$. Then

$$P \left( \binom{x}{k} \right) \leq k \quad \text{implies} \quad \binom{x}{k} < 4^{k + \sqrt{x}} < 4^{k + k^{\frac{3}{4}}} \quad (16)$$

For $k > 1$, we have

$$\frac{4^k}{2\sqrt{k}} < \binom{2k}{k} < \frac{4^k}{\sqrt{2k}} \quad (17)$$
Lower bounds for $\binom{x}{k}$ for $x < k^{3/2}$

- $x \geq 4k$: We have $\left(\frac{4k}{k}\right) \leq \binom{x}{k} < 4^{k+\sqrt{x}} < 4^{k+k^{3/4}}$ and

  $$4^{k+k^{3/4}} > \binom{4k}{k} = \binom{2k}{k} \frac{4k(4k-1) \cdots (3k+1)}{2k(2k-1) \cdots (k+1)}$$

  $$> \frac{4^k}{2 \sqrt{k}} 2^k = \frac{8^k}{2 \sqrt{k}}$$

- $\frac{5}{2}k < x < 4k$: Here $\left(\left\lceil\frac{5}{2}k\right\rceil\right) \leq \binom{x}{k} < 4^{k+\sqrt{x}} < 4^{k+2\sqrt{k}}$ and

  $$4^{k+2\sqrt{k}} > \binom{2k}{k} \left\lceil\frac{5}{2}k\right\rceil \left(\left\lceil\frac{5}{2}k\right\rceil - 1\right) \cdots \left(\left\lceil\frac{5}{2}k\right\rceil - k + 1\right)$$

  $$> \frac{4^k}{2 \sqrt{k}} \left(\frac{5}{4}\right)^k$$

- $2k \leq x \leq \frac{5}{2}k$: Here $\binom{2k}{k} \leq \binom{x}{k} < 4^{k+\sqrt{x}} \leq 4^{k+\sqrt{5k/2}}$ and

  $$4^{k+\sqrt{5k/2}} > \binom{2k}{k} > \frac{4^k}{2 \sqrt{k}}.$$
Comparing the bounds, we obtain for $k \geq 11$ and $x < k^{3/2}$ that $x < 4k$.

Further $k < 103$ for $\frac{5}{2}k < x < 4k$ and $k < 113$ for $2k \leq x \leq \frac{5}{2}k$.

Recall $x < k^{k/\pi(k)}$ and $x < k^2$ for $k \geq 11$ and $x < k^{3/2}$ for $k \geq 37$.

Combining all those: $k \leq 113$;

$x < k^2$ for $11 \leq k \leq 31$; $x < 4k$ for $37 \leq k \leq 113$

and

$x < k^{k/\pi(k)}$ for $k < 11$. 
Idea of Sylvester-Erdős

- Let $S = \{x, x - 1, \ldots, x - k + 1\}$.
- For each prime $p \leq k$, let $0 \leq i_p < k$ be such that $\text{ord}_p(x - i_p)$ is maximal.
- Then for $i \neq i_p$, we have
  \[
  \text{ord}_p(x - i) \leq \text{ord}_p(x - i - (x - i_p)) = \text{ord}_p(i - i_p).
  \]
- Put $S_0 = \{x, x - 1, \ldots, x - k + 1\} \setminus \{x - i_p : p \leq k\}$.
- Then $|S_0| \geq k - \pi(k)$ and
  \[
  \text{ord}_p \left( \prod_{x - i \in S_0} (x - i) \right) \leq \text{ord}_p(i_p! (k - 1 - i_p)!) \leq \text{ord}_p((k-1)!).
  \]
Idea of Sylvester-Erdős

Thus

\[
\left( \prod_{x-i \in S_0} (x - i) \right) \leq \prod_{p < k} p^{\text{ord}_p((k-1)!)} \leq (k - 1)!.
\]

Hence from \(|S_0| \geq k - \pi(k)|\), we get

\[
(x - k + 1)^{k - \pi(k)} \leq (k - 1)! \quad \text{or} \quad x \leq k - 1 + ((k - 1)!)^{\frac{1}{k - \pi(k)}}.
\]

with the equality only when \(k = 2\).

We get \(x \leq 2, 4, 9, 13\) according as \(k = 2, 3, 5, 7\), respectively. This contradicts \(x \geq 2k\). Thus \(k \geq 11\).

For \(11 \leq k \leq 31\), we get \(x \leq 71\).
Finally for $11 \leq k \leq 113$ and

$$x \leq 71 \text{ for } 11 \leq k \leq 31; \ x < 4k \text{ for } 37 \leq k \leq 113$$

we use the following result.

**Lemma 8.**

Let $X$ be a positive real number and $k_0$ be a positive integer. Suppose that $p_{i+1} - p_i < k_0$ for any two consecutive primes $p_i < p_{i+1} \leq p_{\pi(X)+1}$. Then

$$P(x(x-1)\cdot\cdot\cdot(x-k+1)) > k$$

for $2k \leq x \leq X$ and $k \geq k_0$. 
Proof.

Let $2k \leq x \leq X$. We may assume that none of $x, x - 1, \ldots, x - k + 1$ is a prime, since otherwise the result follows. Thus

$$p_{\pi(x)} < x - k + 1 < x < p_{\pi(x)+1} \leq p_{\pi(x)+1}.$$

Hence by our assumption, we have

$$k - 1 = x - (x - k + 1) < p_{\pi(x)+1} - p_{\pi(x)} < k_0,$$

which implies $k - 1 < k_0 - 1$, a contradiction.
Computation gives:

\[ p_{i+1} - p_i < \begin{cases} 
21 & \text{for } p_{i+1} \leq 457 = \pi(452) + 1 \\
8 & \text{for } p_{i+1} \leq 79 = \pi(73) + 1.
\end{cases} \]

Apply Lemma 8 as follows:

- For \( 23 \leq k \leq 31 \), take \( X = 452, k_0 = 21 \);
- For \( 11 \leq k \leq 31 \), take \( X = 73, k_0 = 10 \).

This completes the proof.
Generalization of Sylvester’s Theorem

Recall Sylvester’s result:

\[ P((n+1) \cdots (n+k)) > k \quad \text{for } n \geq k. \]

- Moser: \( P((n+1) \cdots (n+k)) > \frac{11}{10}k \)
- Hanson: \( P((n+1) \cdots (n+k)) > 1.5k \) unless \((n,k) = (3,2), (8,2), (6,5)\).
- Faulkner: \( P((n+1) \cdots (n+k)) > 2k \) if \( n \geq p_{\pi(2k)+1} \).
- For \( k = 2 \), easy to see: \( P((n+1)(n+2)) > 4 \) except when \( n = 1, 2, 7 \).
Laishram and Shorey: We have

\[ P((n + 1) \cdots (n + k)) > 1.95k \quad \text{for } n \geq k. \]

unless \((n, k)\) belongs to an explicitly given finite set.

Here 1.95 cannot be replaced by 2 as

\[ P((k + 1) \cdots (k + k)) < 2k. \]

Laishram and Shorey: We have

\[ P((n + 1) \cdots (n + k)) > 1.8k \quad \text{for } n \geq k. \]

unless \(n - k \in \{3, 4, 5, 8, 11, 13, 14, 18, 63\}\) or

\[(n, k) \in \{(7, 3), (5, 4), (6, 4), (14, 13), (15, 13)\}.\]
Laishram and Shorey: We have

\[ P((n + 1) \cdots (n + k)) > 1.95k \quad \text{for} \quad n \geq k. \]

unless \((n, k)\) belongs to an explicitly given finite set.

Here 1.95 cannot be replaced by 2 as

\[ P((k + 1) \cdots (k + k)) < 2k. \]

Laishram and Shorey: We have

\[ P((n + 1) \cdots (n + k)) > 1.8k \quad \text{for} \quad n \geq k. \]

unless \(n = k \in \{3, 4, 5, 8, 11, 13, 14, 18, 63\}\) or

\((n, k) \in \{(7, 3), (5, 4), (6, 4), (14, 13), (15, 13)\}\).
Laishram and Shorey: We also have

\[ P((n + 1) \cdot \cdots (n + k)) > 2k \] for \( n \geq \max \left( k + 13, \frac{279}{262} k \right) \).

and

\[ P((n + 1) \cdot \cdots (n + k)) > 1.97k \] for \( n \geq k + 13 \).

Observe: 1.97 cannot be replaced by 2 since there are arbitrary long chains of consecutive composite positive integers.

The assumption \( n \geq \frac{279}{262} k \) is necessary since \( P((n + 1) \cdot \cdots (n + k)) \leq 2 \times 262 \) when \( n = 278 \) and \( k = 262 \).
Generalization for arithmetic progression

Let $n, d$ be positive coprime integers with $d > 1$ and put

$$\Delta(n, d, k) = n(n + d)(n + 2d) \cdots (n + (k - 1)d).$$

Observe; $P(\Delta(n, d, 2)) = 2$ if and only if $n = 1$, $d = 2^r - 1$ with $r > 1$.

Sylvester: $\Delta(n, d, k) > k$ if $n \geq d + k$.

Let $d = 2$. Then

$$\Delta(n, 2, k) = n(n + 2)(n + 2d) \cdots (n + 2(k - 1)).$$

Let $n \leq k$. Then $P(\Delta(n, 2, k)) < 2k$ implies

$P((n + k) \cdots (n + 2k - 1)) \leq 2k.$

This gives $n + k - 1 < \max(k + 13, \frac{279}{262}k)$ or $n < \max(14, \frac{17}{262}k + 1)$.

We consider $k > 2$ and $d > 2.$
Generalization for arithmetic progression

- **Langevin:** $P(\Delta(n, d, k)) > k$ if $n > k$.
- **Shorey and Tijdeman:** $P(\Delta(n, d, k)) > k$ unless $(n, d, k) = (2, 7, 3)$ where $P(2 \cdot 9 \cdot 16) = 3$.
- **Laishram and Shorey:** For $k > 2$ and $d > 2$, we have
  
  $$P(n(n + d) \cdots (n + (k - 1)d)) > 2k$$

  unless $(n, d, k)$ is given by

  - $k = 3, n = 1, d = 4, 7$;
  - $n = 2, d = 3, 7, 23, 79$;
  - $n = 3, d = 61; n = 4, d = 23$;
  - $n = 5, d = 11; n = 18, d = 7$;
  - $k = 4, n = 1, d = 3, 13; n = 3, d = 11$;
  - $k = 10, n = 1, d = 3$.

- This is best known result.
Laishram and Shorey: $P(\Delta(n, 3, k)) > 3k$ when $n > 3k$ and $(n, k) \neq (125, 2)$.

Laishram and Shorey: $P(\Delta(n, 4, k)) > 4k$ when $n > 4k$ and $n \neq 21, 45$ when $k = 2$.

Conjecture: Fix $d$. For $n > dk$ and $k > 1$, we have

$$P(n(n + d) \cdots (n + (k - 1)d)) > dk$$

except for finitely many $n$ and $k$.

This is open.
Though we have an elementary proof of Sylvester’s result for consecutive integers, the proof of other results need advanced mathematical ideas from Prime number theory, linear forms in logarithms, Thue equations etc.

However many of the ideas used in the elementary proof can be generalised and are used in subsequent proofs.
Let $\omega(m)$ denote the number of distinct prime divisors of $m$. For example, $\omega(10) = 2$.

The result of Sylvester is in fact

$$\omega(n(n + 1) \cdots (n + k - 1)) > \pi(k) \quad \text{if } n > k.$$ 

$\omega(n(n + 1)) \geq 3$ except when $n = 2^p - 1$, a *Mersenne Prime* or $n = 2^{2^r} + 1$, a *Fermat Prime* or $n \in \{1, 2, 3, 8\}$.

In fact the famous Catalan Conjecture, now a theorem, states that $x^r - y^s = 1$, with $x > 1, y > 1, r > 1, s > 1$. 

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In fact the famous Catalan Conjecture, now a theorem, states that $x^r - y^s = 1$, with $x > 1, y > 1, r > 1, s > 1$.
Let \( k \geq 3 \). Observe that

\[
\omega(n(n+1) \cdots (n+k-1)) = \pi(2k) \quad \text{when } n = k + 1.
\]

If \( k + 1 \) is prime and \( 2k + 1 \) is composite, then we observe that

\[
\omega((k+2) \cdots (2k+1)) = \pi(2k) - 1.
\]

Let \( k + 1 \) be a prime of the form \( 3r + 2 \). Then
\( 2k + 1 = 3(2r + 1) \) is composite. Since there are infinitely many primes of the form \( 3r + 2 \), we see that there are infinitely many \( k \) for which \( k + 1 \) is prime and \( 2k + 1 \) is composite.

Therefore \( \omega(\Delta(n, k)) \geq \pi(2k) - 1 \) is valid for infinitely many \( k \). Thus an inequality sharper than
\( \omega(\Delta(n, k)) \geq \pi(2k) - 1 \) for \( n > k \) is not valid.
Number of prime divisors

- **Saradha and Shorey:**
  
  \[ \omega(n(n+1) \cdots (n+k-1)) \geq \pi(k) + \left\lfloor \frac{1}{3}\pi(k) \right\rfloor + 2 \quad \text{if } n > k \]

  unless \((n, k)\) belongs to a finite set.

- **Laishram and Shorey:** For \(n > k\), we have
  
  \[ \omega(n(n+1) \cdots (n+k-1)) \geq \pi(k) + \left\lfloor \frac{3}{4}\pi(k) \right\rfloor - 1 + \delta(k) \]

  unless \((n, k)\) belongs to a finite set and

  \[
  \delta(k) = \begin{cases} 
  2 & \text{if } 3 \leq k \leq 6 \\
  1 & \text{if } 7 \leq k \leq 16 \\
  0 & \text{otherwise.}
  \end{cases}
  \]
Laishram and Shorey: Let $n > k$. Then

$$\omega(n(n + 1) \cdots (n + k - 1)) \geq \min \left( \pi(k) + \left\lfloor \frac{3}{4} \pi(k) \right\rfloor - 1 + \delta(k), \pi(2k) - 1 \right).$$

Laishram and Shorey: Let $(n, k) \neq (6, 4)$. Then

$$\omega(n(n + 1) \cdots (n + k - 1)) \geq \pi(2k) \quad \text{for} \quad n > \frac{17}{12} k.$$

For $n > k$, we cannot have $\pi(2k)$ or $2\pi(k)$ as a lower bound.
Laishram and Shorey: Let $\epsilon > 0$ and $n > k$. Then there exists a computable number $k_0$ depending only on $\epsilon$ such that for $k \geq k_0$, we have

$$\omega(n(n + 1) \cdots (n + k - 1)) \geq (2 - \epsilon)\pi(k).$$
Grimm’s Conjecture

Suppose $n + 1, \cdots, n + k$ are all composite numbers, then there are distinct primes $P_j$ such that $P_j | (n + j)$ for $1 \leq j \leq k$.

This is open and considered quite difficult.

This implies: If $n, n + 1, \cdots, n + k - 1$ are all composite, then $\omega(n(n + 1) \cdots (n + k - 1)) \geq k$ which is also open.

Grimm’s Conjecture has been verified for all $n \leq 1.9 \times 10^{10}$ and for all $k$. 

Shanta Laishram  
Prime divisors in an arithmetic progression
Schinzel’s Hypothesis H:

Let $f_1(x), \ldots, f_r(x)$ be irreducible non constant polynomials with integer coefficients and the leading coefficients positive. Assume that for every prime $p$, there is an integer $a$ such that the product $f_1(a) \cdots f_r(a)$ is not divisible by $p$. Then there are infinitely many positive integers $m$ such that $f_1(m), \ldots, f_r(m)$ are all primes.

This is open. Considered quite difficult.

Schinzel’s Hypothesis H implies that

$\omega(\Delta(n, d, k)) = \pi(k)$, $\pi(k) = 1$ for $k = 3$ and $k \in \{4, 5\}$, respectively for infinitely many pairs $(n, d)$.
Shorey and Tijdeman: $\omega(\Delta(n, d, k)) \geq \pi(k)$

Moree: $\omega(\Delta(n, d, k)) > \pi(k)$ for $k \geq 4$ and $(n, d, k) \neq (1, 2, 5)$.

Saradha, Shorey and Tijdeman: For $k \geq 6$,

$$\omega(\Delta(n, d, k)) > \frac{6}{5} \pi(k) + 1$$

unless $(n, d, k)$ belongs to

$$\{(1, 2, 6), (1, 3, 6), (1, 2, 7), (1, 3, 7), (1, 4, 7), (2, 3, 7), (2, 5, 7), (3, 2, 7), (1, 2, 8), (1, 2, 11), (1, 3, 11), (1, 2, 13), (3, 2, 13), (1, 2, 14)\}.$$
Laishram and Shorey: For $k \geq 9$

$$\omega(\Delta(n, d, k)) \geq \pi(2k) - \rho$$

unless $n, d, k$ is given by

$$\begin{cases} 
  n = 1, \ d = 3, \ k = 9, 10, 11, 12, 19, 22, 24, 31; \\
  n = 2, \ d = 3, \ k = 12; \ n = 4, \ d = 3, \ k = 9, 10; \\
  n = 2, \ d = 5, \ k = 9, 10; \ n = 1, \ d = 7, \ k = 10.
\end{cases}$$

where

$$\rho = \rho(d) = \begin{cases} 
  1 \text{ if } d = 2, \ n \leq k \\
  0 \text{ otherwise.}
\end{cases}$$
Laishram and Shorey: Let \( k \geq 4 \). Then

\[
\omega(\Delta(n, d, k)) \geq \pi(2k) - 1
\]

except at \((n, d, k) = (1, 3, 10)\).

This proves a conjecture of Moree.
Binary Recurrence Sequences

Exponential Diophantine Equations

A weaker Conjecture: Fix $d$. For $n > dk$ and $k > 1$, we have

$$P(n(n + d) \cdots (n + (k - 1)d)) > \frac{dk}{200}.$$ 

This conjecture implies a conjecture of Erdos which states that the equation

$$n(n + d) \cdots (n + (k - 1)d) = y^2$$

has no solution in positive integers $n, d, k$ with $\gcd(n, d) = 1$ and $k \geq 2$. 
Irreducibility of Polynomials

For example, the *Truncated Exponential Polynomials*

\[ E_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \]

are irreducible for all \( n \geq 1 \).
Thank you