Ramanujan's Tau function and Modular forms

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Totally multiplicative functions are multiplicative functions that satisfy

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for any pair of integers (m, n) that are greater than 0.

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- If p is a prime number, then $\sigma(p) = \phi(p) + \tau(p)$. For instance, $p = 7$ then $\phi(p) = p - 1 = 6, \tau(p) = 2, \sigma(p) = 8$.

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Euler's identity:

$$\zeta(s) = \prod_{p \ prime} \frac{1}{1 - p^{-s}} = \prod_{p \ prime} \frac{1}{1 - \frac{1}{p^s}}. \tag{1}$$

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Figure: Bernhard Riemann (German Mathematician) 1826-1866

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Figure: Leonhard Euler (Swiss Mathematician) 1707-1783

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Note that Euler's identity converts an "infinite sum" into an "infinite product"!

Ramanujan's Tau function

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This is an arithmetic function, defined on positive integers

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n,$$

where $q \in \mathbb{C}, |q| < 1$, ("$q = e^{2\pi iz}$"), $z \in \mathcal{H}$, where $\mathcal{H}$ is the upper half plane $\{(x+iy)|x = 0, y > 0, y \in \mathbb{R}\}$.

Srinivasa Ramanujan

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Figure: Srinivasa Ramanujan 1887 - 1920

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- Mordell proved multiplicativity in 1917.

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- Deligne proved the second conjecture above in 1976.

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Figure: Louis Mordell 1888 - 1972

Pierre Deligne

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Figure: Pierre Deligne 1944 -

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Fact:

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sum_{k=1}^{n-1}\sigma_5(k)\sigma_5(n-k).$$

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- f is holomorphic as $z \to i\infty$.

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$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

It also has an expression as an infinite product

$$L(\Delta, s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_{p \ prime} \left(1 - \frac{\tau(p)}{p^s} + \frac{1}{p^{s-11}}\right)^{-1}.$$

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More than half a dozen top prizes (Fields Medal, Abel Prize) have been inspired by Ramanujan's work.

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This is again an instance of Ramanujan's gift of divination, not only that he foresaw or unveiled hitherto unknown fields, but most of his conjectures lead back to sources of great depth and abundant yield.

Hans Rademacher (1892 - 1969)

Thank you!