GONITSORA

SEPTEMBER 13, 2020.

RANDOM WALK   an idea

B. V. Rao
CHENNAI MATHEMATICAL INSTITUTE

Shall discuss a probability model.
J L DOOB famous probabilist once said:

The fact that nonprobabilists commonly denote functions by f, g, and so on whereas probabilists tend to call functions random variables and use the notation X, Y and so on at the other end of the alphabet helped to make nonprobabilists suspect that mathematical probability was hocus pocus rather than mathematics.

And the fact that probabilists called some integrals “expectations” and used the letters E or M instead of integral signs strengthened the suspicion.

probability is more than just calcualting ratios/probabilities. You build models for practical phenomena.

Shall start with one simple example.

Imagine following experiment on integers

$$Z = \{\ldots\ldots, -2, -1, 0, +1, +2, \ldots, \ldots\}$$

Have a fair coin. I start at 0. Toss the coin,
Heads up move one step forward,
Tails up, move one step backward.
CONTINUE FOREVER. This is the game.

X_n position at time n.

Thus $X_0 = 0$.
$X_1 = +1$ probability $1/2$; $X_1 = -1$ probability $1/2$

$X_2 = +2$ probability $1/4$ (HH)
$X_2 = -2$ probability $1/4$ (TT)
$X_2 = 0$ probability $1/2$ (HT/TH)

Question: is there a chance of returning to zero?
Answer: Yes with chance at least $1/2$.

Question: Are you sure to return to zero?

$$C = \{X_n = 0, \text{ for some } n \geq 1\}$$

Question: $P(C) = 1$ or $P(C) < 1$

Depends on infinitely many variables. So not easy to answer
For each n easy to calculate
$p_n = P(X_n = 0)$ Clearly, $p_0 = 1$

$$p_{2n+1} = 0, \quad p_{2n} = \binom{2n}{n}\frac{1}{2^{2n}}$$

These events definitely make up C but are not disjoint.
So can not add up. Disjointify
$f_1 = P(X_1 = 0)$

$f_2 = P(X_1 \neq 0, X_2 = 0)$

$f_3 = P(X_1 \neq 0, X_2 \neq 0, X_3 = 0)$

$f_4 = P(X_1 \neq 0, X_2 \neq 0, X_3 \neq 0, X_4 = 0)$

and so on [For odd n above event is empty. Do not get distracted]

$$f_n = P(X_1 \neq 0, \cdots, X_{n-1} \neq 0;\ X_n = 0) \quad f_0 = 0.$$

Question: Is $\sum f_n = 1$?

These f_n are difficult to calculate. (in this example you can)

Mathematicians found a good way of remembering sequence of numbers
Generating Functions

$$P(s) = \sum_{n \geq 0} p_n s^n; \quad F(s) = \sum_{n \geq 0} f_n s^n \quad \text{for} \quad 0 \leq s < 1.$$

These Converge at least in the range shown.
Since Coefficients are non-negative

$$\lim_{s \uparrow 1} F(s) = \sum f_n; \qquad \lim_{s \uparrow 1} P(s) = \sum p_n.$$

Question: $\lim_{s \uparrow 1} F(s) = 1$ or < 1?

Simple Powerful relation called Renewal equation.

$$\text{For } n \geq 1; \quad p_n = \sum_{k=0}^{n} f_k \; p_{n-k}. \tag{1}$$

NOT true for $n = 0$, $p_0 = 1$ but $f_0 p_0 = 0$.

Let $A = (X_n = 0)$ $p_n = P(A)$.

$A_1 = (X_1 = 0, X_n = 0)$ Probability: $f_1 p_{n-1}$

$A_2 = (X_2 = 0, X_1 \neq 0, X_n = 0)$ probability: $f_2 p_{n-2}$

$A_3 = (X_3 = 0, X_2 \neq 0, X_1 \neq 0, X_n = 0)$ Probability: $f_3 p_{n-3}$

finally $A_n = (X_n = 0, X_{n-1} \neq 0, X_{n-2} \neq 0, \ldots, X_1 \neq 0\}$ $f_n p_0$

A is disjoint union of these sets. Probabilities add up. Done.

[you can ignore A_k for odd values of k]

Cauchy theorem: Have convergent series of non-negative terms

$$\sum_{n\geq 0} a_n = a \qquad \sum_{n\geq 0} b_n = b$$

Make new series $\sum_{n\geq 0} c_n$ $\qquad c_n = \sum_{k=0}^{n} a_k b_{n-k}$. $\quad$ Then $\sum c_n = ab$.

Fix $0 \leq s < 1$. Take $a_n = f_n s^n$ and $b_n = p_n s^n$
$c_n = b_n$ for $n \geq 1$ by Renewal equation. $c_0 = 0, b_0 = 1$

$P(s) - 1 = P(s)F(s)$ $\qquad$ or $P(s) = \frac{1}{1-F(s)}$.
consequence
$\lim_{s\uparrow 1} P(s) = \infty \leftrightarrow \lim_{s\uparrow 1} F(s) = 1$

$$\sum p_n = \infty \leftrightarrow \sum f_n = 1$$

Thus we are sure to return to the origin if $\sum p_n = \infty$

chances of returning to the origin is < 1 if $\sum p_n < \infty$.

extremely pleasing for two reasons.
One: p_n easy to calculate, as we saw.

Two: need not know exact value of $\sum p_n$
Only finite or not.

UPSHOT: Sure to return iff $\sum \binom{2n}{n} \frac{1}{2^{2n}} = \infty$.

Stirling formula.

$$n! \sim \sqrt{2\pi} e^{-n} n^{n+\frac{1}{2}}; \quad i.e. \quad \lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}\ e^{-n}\ n^{n+\frac{1}{2}}} = 1.$$

$$\frac{(2n)!}{n!n!} \sim \frac{\sqrt{2\pi}\ e^{-2n}\ (2n)^{2n+\frac{1}{2}}}{\sqrt{2\pi}\ e^{-n}\ n^{n+\frac{1}{2}} \sqrt{2\pi}\ e^{-n}\ n^{n+\frac{1}{2}}}$$
$$= \frac{1}{\sqrt{\pi}}\ \frac{1}{\sqrt{n}} 2^{2n}$$

So

$$p_{2n} = \binom{2n}{n} \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi}}\ \frac{1}{\sqrt{n}}$$

LO and BEHOLD: We are sure to return.

(a_n) and (b_n) strictly positive sequences, say
$(a_n) \sim (b_n)$ if $a_n/b_n \to 1$.
[Not same as saying $a_n - b_n \to 0$ $a_n = n^2 + n;\ b_n = n^2$]

(1) In such a case, after some stage $\frac{1}{2} \ \le \ \frac{a_n}{b_n} \ \le \ \frac{3}{2}$.

$$\frac{1}{2}\, b_n \ \le \ a_n \ \le \ \frac{3}{2}\, b_n.$$

As a consequence, $\sum a_n$ and $\sum b_n$ both diverge or both converge.

(2) $(a_n) \sim (b_n)$ and $(c_n) \sim (d_n)$ then $(a_n c_n) \sim (b_n d_n)$ and also $(a_n/c_n) \sim (b_n/d_n)$.

for ex:

$$\frac{a_n c_n}{b_n d_n} = \frac{a_n}{b_n}\frac{c_n}{d_n} \to 1$$

Called Simple Symmetric Random Walk.
Simple: moving only one unit at a time
symmetric: right and left equal probability

Simple proof of above can also be given
Can ask many questions:
Howlong does it take to return to zero? Expected time
How long does it take to reach 200000 etc

Mathematics like Music allows Improvizations

We see several useful improvizations of this idea rather than just getting deep into this one problem discussed above.

(A) Why only Z?
can extend to two or three dimensions

From (i,j) go to $(i+1,j)$, $(i-1,j)$, $(I,j+1)$, $(i,j-1)$ each with probability $1/4$.
Start at $(0,0)$ Even Now $p_{2n+1}=0$

$$p_{2n}=\sum_{m=0}^{n}\frac{(2n)!}{m!m!(n-m)!(n-m)!}\frac{1}{4^{2n}}$$

$$=\binom{2n}{n}\sum_{m=0}^{n}\left[\frac{n!}{m!(n-m)!}\right]^2\frac{1}{4^{2n}}$$

$$=\binom{2n}{n}\binom{2n}{n}\frac{1}{2^{2n}}\frac{1}{2^{2n}}\sim\frac{1}{\pi}\frac{1}{n}$$

In three dimensions: From (i, j, k) you go to the six neighbours

$(i+1, j, k)$, $(i-1, j, k)$, $(i, j+1, k)$, $(i, j-1, k)$, $(i, j, k+1)$, $(i, j, k-1)$

Can show

$$p_{2n} \sim C \frac{1}{n^{3/2}}$$

There is a chance of never returning!

Of course there are several other interesting questions.

(B) Another look at RW.
Can think of RW sums of iid variables:

What was RW?
Take independent random variables $\{\epsilon_n, n \geq 1\}$ all with same distribution
$\epsilon_1 = \pm 1$ probabilities $1/2$. Easy to see Random walk is nothing but

$$X_n = x + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n; \quad X_0 = x$$

(if you start from x)
Take any random variables $\{\epsilon_n, n \geq 1\}$ all having the same distribution NOT necessarily above BUT independent
Put $X_0 = x$, $X_n = x + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ as above.
This is called general random walk on R.

For example:
ϵ_1 takes values $\{0, \pm 1, \pm 2, \pm 3\}$ each with probability $1/7$.
Or $\epsilon_1 \sim N(0,1)$. That is

$$P(a < \epsilon_1 \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Reduces to sums of i.i.d. random variables. Huge history.
Very useful in practice.
i.i.d. random variables have the interpretation of random sample from a population.

($\bullet$) Weak Law of Large Numbers
(X_n) i.i.d. mean μ, finite variance then

$$P(|\frac{X_1 + \cdots + X_n}{n} - \mu| > \epsilon) \to 0, \quad \forall \epsilon > 0$$

($\bullet\bullet$) Strong Law of Large Numbers
(X_n) i.i.d. mean μ

$$\frac{X_1 + \cdots + X_n}{n} \to \mu \quad \text{almost surely}$$

($\bullet \bullet \bullet$) Central Limit Theorem
(X_n) i.i.d. mean μ variance σ^2

$$P(a < \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq b) \to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

(C) DIFFERENT LOOK: RW is on integers or reals or $\mathbb{Z}^d$, $\mathbb{R}^d$

Why hang on to integers or reals?

Take a finite group G. Take elements, say, g, h, g^{-1}, h^{-1} from the group. Start from any point in G.
When at $x \in G$ move to
xg, xg^{-1}, xh, xh^{-1} each with probability $1/4$. Continue.

This is an example of Random walk on the group G.
The kind of questions you ask are different depending on the problem at hand.

Instead of abstract discussion we see a specific example soon.

(D) YET ANOTHER LOOK at RW

Why group? Take any set S (finite or not)
Just like in Physics: Prescribe law of motion and how to start.
Ask questions about the motion.
To illustrate Take a finite set S (countable set is also fine)

Take a stochastic matrix: $P = ((p_{ij}))$ $(i, j \in S)$
This means $p_{ij} \geq 0$ for all i, j. $\sum_j p_{ij} = 1$ for all i.

points of S are called states.
START from any state i^*.
If at x move to y with probability p_{xy}. Continue.

called MARKOV CHAIN with state space S, transition matrix P, initial distribution μ.

Has huge theory and very useful. Here is an EXAMPLE

Take a deck of cards.

Arrange in a stack

$Q\heartsuit$	1
$A\diamondsuit$	2
$3\spadesuit$	3
$\vdots$	$\vdots$
$\vdots$	$\vdots$
$9\clubsuit$	51
$7\diamondsuit$	52

Can Start with any arrangement.

Rule: pick at random $1 \leq i \leq 52$, $1 \leq j \leq 52$
interchange cards at places i and j in the stack.
Of course, if $i = j$, then stack remains same.
Continue.
Here S: set of all posible arrangements of the cards.
In the long run all 52! arrangements are equally likely.

What does this mean?
Let X_n be the arrangement after n interchanges. Of course it is random, depends on what i's and j's were selected.
For any given arrangement π, $P(X_n = \pi) \to \frac{1}{52!}$

OR
Rule: Pick $1 \leq i \leq 52$ at random
Take top card. put it at a random place i.
Of course if $i = 1$, stack remains same.
Same final result: All arrangements are equally likely.
Thus here is a way of selecting at random one among the 52! permutations.
This is very important in computer science/physics and so on
Sometimes you do not even have a full list of elements of your set.
You know it is finite. Want to pick one at random.

Another model: Have LOTS of balls. Have a box.
Have coin, chance of heads p, $0 < p < 1$. Have a number $\lambda > 0$
Everyday shall do the following:

For each ball in the box, toss coin.
Heads up remove Tails up keep in the box.

AND independently

add certain number of balls. How many?
k with probability $e^{-\lambda}\,\lambda^k/k!$; $\quad k = 0, 1, 2, 3 \ldots\ldots$
This is a Markov Chain

What happens in the long run?
$P(X_n = k) \to e^{-\lambda/p}\,(\lambda/p)^k/k!$

Another model: Have LOTS of balls. Have a box.
Have coin, chance of heads p, $0 < p < 1$. Have a number $\lambda > 0$
Everyday shall do the following:

For each ball in the box, toss coin.
Heads up remove Tails up keep in the box.

AND independently

add certain number of balls. How many?
k with probability $e^{-\lambda}\,\lambda^k/k!$; $\quad k = 0, 1, 2, 3 \ldots\ldots$
This is a Markov Chain

What happens in the long run?
$P(X_n = k) \to e^{-\lambda/p}\,(\lambda/p)^k/k!$

Actually I have no balls no box
neither removing nor adding balls everyday
Then what?

Have a solution containing some pollen particles.
Balls are pollen particles
Have a geometrically well defined volume V in the solution.
This region V is the Box
Time period not a day but few hundredths of a second
Each time period some particles enter this region V and certain particles in this region V leave
Trying to understand what happens in the long run

AStrophysicist S. Chandrasekhar

R J Taylor (Astronomer) says: Chandrasekhar was a classical applied mathematician whose research was primarily applied in Astronomy and whose like will probably be never seen again.

(E) Yet another look at RW

Observe that for Random walk $(X_n, n \geq 0)$

Given $X_{n-1} = a$, X_n takes values $a \pm 1$ with equal probability,

So (conditional) expectation is a. This is expressed

$E(X_n \| X_0 = a_0, X_1 = a_1, \ldots, X_{n-1} = a) = a$

Or

$E(X_n \| X_0, \ldots, X_{n-1}) = X_{n-1}$

Sequence of random variables with this property MARTINGALE

Very very useful. For example, in finance, Prices fluctuate,

X_n price (of a particular commodity) on day n

Could any sequence (X_n) be thought of as model?

'No Arbitrage' forces martingale or nearly a martingale (equivalent probability) for finite horizon market model.

(F) FINAL Look: More fundamental question
why not move continuously? HOW?

Our understanding of continuous things is ONLY through discrete approximations.
Came across this in High School, thanks to Newton.

Particle moving. At time t it is at $x(t) = t^2$. (Fix $0 \leq t \leq 1$).
What is its velocity? Velocity=distance/time

It is continuously changing. We do not understand velocity at a specific time.

Consider times $t = 0, \quad 1/n, \quad 2/n, \quad 3/n, \ldots 1$.
During k/n to $(k+1)/n$ distance travelled $x(\frac{k+1}{n}) - x(\frac{k}{n})$

velocity = distance/time $=[x(\frac{k+1}{n}) - x(\frac{k}{n})] \ / \ [1/n]$.

Take smaller partition etc Finally arrive at concept of 'velocity' at a time t.

Execute RW more frequently, with smaller jumps. For example.

Move at time units $\frac{1}{n}, \quad \frac{2}{n}, \quad \frac{3}{n}, \quad \ldots$

every time unit $\frac{1}{n}$, a jump $\pm\frac{1}{\sqrt{n}}$

Time $\frac{k}{n}$ if at $\frac{m}{\sqrt{n}}$ Then at time $\frac{(k+1)}{n}$ go to $\frac{m+1}{\sqrt{n}}$ or $\frac{m-1}{\sqrt{n}}$

If this has a limit it can be regarded as continuous motion.

Limit in what sense? Does it exist? What is this 'coupling' of $1/n$ with $1/\sqrt{n}$?
Leads to nice story. Leads to
BROWNIAN MOTION

What is this $1/n$ (duration) and $\pm 1/\sqrt{n}$ (displacement) business?

To get meaningful limits you need some balancing act.
Shall explain with a simple college example. Compound Interest

Invest 1Re. Bank says interest rate: $r > 0$ (per Rupee per year)
Year End: $(1 + r)$ year = one unit of time.

Bank says: every $1/n$ unit of time, we update with interest of r/n.
Year End: $(1 + \frac{r}{n})^n$. unit of time: $1/n$ year

Bank says interest accrues continuously? What does it mean?
Year End: $\lim(1 + \frac{r}{n})^n = e^r$

Any continuous 'something' is difficult to comprehend; it is limit of discrete things.

Instead consider two scenarios

Bank says: every $1/n$ unit of time we update with interest $\frac{r}{n^{1.01}}$.
Year end: $(1 + \frac{r}{n^{1.01}})^n$
interest accrues continuously.
Year End: $\lim_n (1 + \frac{r}{n^{1.01}})^n$ Will you accept?

Bank says every $1/n$ unit of time we update with interest $\frac{r}{n^{0.99}}$.
Year end: $(1 + \frac{r}{n^{0.99}})^n$
interest accrues continuously.
Year End: $\lim_n (1 + \frac{r}{n^{0.99}})^n$ Can the bank afford to do that?

Instead consider two scenarios

Bank says: every $1/n$ unit of time we update with interest $\frac{r}{n^{1.01}}$.
Year end: $(1+\frac{r}{n^{1.01}})^n$
interest accrues continuously.
Year End: $\lim_n (1+\frac{r}{n^{1.01}})^n$ Will you accept?

Bank says every $1/n$ unit of time we update with interest $\frac{r}{n^{0.99}}$.
Year end: $(1+\frac{r}{n^{0.99}})^n$
interest accrues continuously.
Year End: $\lim_n (1+\frac{r}{n^{0.99}})^n$ Can the bank afford to do that?

Limit in First case ONE. Second Case: INFINITY.
So meaningful power of $1/n$ is 1
In continuous RW meaningful power of $1/n$ is 0.5

Statisticians J Neyman and E L Scott say:

Each attempt to treat mathematically a complicated category of phenomena must rely on idealizations of certain factors deemed of predominant importance and must ignore innumerable other factors. Whether, as a whole the model is close enough to the actual phenomenon to be useful for practical purposes, for example, prediction, can be established only through comparison with data. However, even if a proposed model proves totally inadequate, it is hoped that the mere process of establishing the model's inadequacy will contribute to a better understanding of the fascinating phenomenon which is being modelled.

THANK YOU