Classification Problems in Mathematics
Illustrative Examples

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Introduction

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What does it mean to say 'classify'? It depends on the particular property one is studying. I shall give examples to illustrate this.

1. Prime Numbers

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Take any number of primes, say $\{p_1, \cdots, p_r\}$. Then any prime divisor of the number $p_1. \cdots p_r + 1$ is not any of the p_i's, and hence there is at least one more prime.

2. Twin primes

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There has been a lot of progress on this question but the original problem is still to be answered! So we may ask a weaker question: Is there some integer c such that there are infinitly many pairs of primes whose difference is at most c? There has been a lot of progress in this direction in this century.

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In the 21-st century, it has been shown that one can take $c = 246$. Number theorists are optimistic that this can be reduced (in finite time) to 2.

3. Groups

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In undergraduate classes, one learns that the alternate groups in n letters are simple if (and only if) n is other than 4. Moreover all groups of prime order are simple and are the only simple abelian groups. The classification of finite non-abelian simple groups was completed towards the end of the last century and I will return to it shortly.

4. Representations of finite groups

One would like to classify all representations of a given finite group G by linear transformation of a (finite dimensional) vector space over $\mathbb{C}$. It is easy to see that any such representation is actually by unitary transformations with respect to a suitable hermitian inner product. This implies that it is enough to classify *irreducible representations***.**

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In the case of the symmetric groups S_n however, the conjugacy classes of elements is in natural bijection with set of partitions of n. One can also associate to each partition a representation and this leads to an important study introduced by Young.

5. Tori and their representations

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If $G = T = S^1$ then every irreducible representation is 1-dimensional (in other words, a character) and characters are given by $z \to z^n, n \in \mathbb{Z}$. Thus the group of characters is simply $\mathbb{Z}$. If functions on T are considered as periodic functions on $\mathbb{R}$, functions on $\mathbb{Z}$ may be thought of as Fourier coefficients. Thus this study is the same as the theory of Fourier series. This works in the more general context, i.e. when G is an r-dimensional torus T^r.

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A) with complex coefficients which are unitary, $n \geq 2$;
B) which have real coefficients and are orthogonal, n odd and ≥ 3,
C) which are unitary and are symplectic, $n \geq 6$;
D) which have real coefficients and are are orthogonal with n even ≥ 8.
These are called *classical groups*. Apart from these, there are five groups, called *exceptional groups*.

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These are called *classical groups*. Apart from these, there are five groups, called *exceptional groups*.

These essentially classify all compact, connected simple Lie groups.

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Next question is: Can we classify their representations? In fact, Every such group contains a maximal dimensional torus, say T, and any representation of G is determined by its restriction to T.

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It is a nice formula, but it is difficult to explicitly write down these multiplicities. Seshadri studied this question and the final answer was given by Littelmann in terms of what became known as the *Lakshmibai-Seshadri paths.*

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We then would like to study the representations of the non-compact real forms.

What kind of representations should one study? Since the study of unitary characters of $\mathbb{R}$ leads to Fourier transforms, we may first look at unitary representations of real simple Lie groups.

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This classification was the monumental work of Harishchandra. Firstly, any unitary representation is a direct integral of irreducible unitary representations. In order to classify irreducible representations one constructs a measure space and to every point one associates a unitary irreducible representation.

9. Topological Classification of surfaces

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They are determined upto homeomorphism by their fundamental groups. Their fundamental group can be explicitly described. Take a free group with $2g$ generators $\{a_i, b_i\}, 1 \leq i \leq g$, say. Take the quotient $\pi(g)$ of this free group by the normal subgoup generated by the element $\Pi(a_i b_i a_i^{-1} b_i^{-1})$. The fundamental group of the surface is one such.

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The integer g is called the *genus* of the surface X. Thus the surface is topologically classified by its genus g.

Genus 0 and Genus 1

Firstly, if $g = 0$, we have $\pi(X) = (1)$, that is to say the surface is simply connected. In fact, the surface is just the sphere S^2.

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More generally, If $g \geq 0$, it is the surface with g holes'.

Riemann sphere and elliptic curves

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When $g = 1$, it can be provided with a complex structure in many ways. From the complex analytic point of view, these are quotients of $\mathbb{C}$ by a lattice, and so functions on them may be thought of as *doubly periodic* **functions. While all these surfaces are tori topologically, they are not all isomorphic as complex analytic objects.**

10. Classification of Elliptic Curves

Given a lattice L in $\mathbb{C}$, Weierstrass constructed the simplest possible non-constant meromorphic function on $\mathbb{C}$ invariant under translations by elements of L or, what is the same, a meromorphic function on the quotient $E = \mathbb{C}/L$. It is called *Weierstrass function* **and is denoted $\wp$.**

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Indeed, E itself is a curve in the projective plane P^2 given by the homogeneous equation $y^2z = 4x^3 - g_2z^2 - g_3z^3$. where g_2 and g_3 are constants determined by the lattice.

Thus the study of Riemann surfaces of genus 1 and the study of (smooth) cubic curves in the projective plane coincide. Many complex analytic questions on such a surface and geometric properties of cubic curves are closely related.

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From the geometric view=point, these are called *elliptic curves,* The complex number $\Delta(E) = g_2^3 - 27g_3^2$ is called the discriminant of E and its non-vanishing ensures that E is singularity-free. The complex number $j = g_2^3/\Delta$ is called the *modulus* of E and associating j to E gives a bijection of the set of isomorphism classes of elliptic curves into $\mathbb{C}$.

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Thus this classification becomes algebraic geometric!

11. Riemann surfaces of higher genus

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Again it is true that all of them can be considered as curves, given by homogeneous polynomials in a projective space. The set of all such Riemann surfaces is parametrised by a $3g - 3$ dimensional complex space, called *the moduli space of Riemann surfaces of genus g.* Actually one took a Riemann surface together with a homeomorphism with a surface of genus g and constructed a moduli space of such objects with a suitable notion of isomorphism.

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Such a moduli spae is called a *Teichmuller space* **after a German mathematician called Teichmuller. The study of this moduli space occupied several decades. Similar questions of classification of higher dimensional manifolds, from several points of view, has been the focus of much research in the second half of the twentieth century.**

Meromorphic functions on a Riemann surface

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In other words, if we are given a set of points p_i with multiplicites l_i and a set of points q_j with multiplicities m_j, one wants to know if there exists a meromorphic function with zeros at p_i with multiplicities l_i and poles at q_j with multiplicities m_j.

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We may formulate the question as follows. Take an element $D = \sum n_i x_i$ of the free abelian group over X. It is called a *divisor* in X. We can obviously associate to any meromorphic function f such an element denoted $div(f)$. Conversely given such a D, when is it the divisor of a meromorphic function?

Divisor classes

On the Riemann sphere, the necessary and sufficient condition for a divisor $\sum n_i x_i$ to be the divisor of a meromorphic function is that $\sum n_i = 0$. In general, the integer $\sum n_i$ is called the *degree* of the divisor.

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If $g > 0$, the vanishing of the degree of a divisor is a necessary condition for it to be assosiated to a meromorphic function, but not sufficient.

We say that two divisors are *equivalent* if their difference is the divisor associated to a meromorphic function. So the question can now be formulated as classifying divisors of a given degree d upto this equivalence.

12. Jacobian

Jacobi studied this question. The equivalence classes of all divisors of a given degree d can be provided with the structure of a complex manifold, called *Jacobian* J^d of X. Since X can be provided with the structure of a projective curve, it is a good question if J^d can be provided with the structure of a projective algebraic variety. Indeed it can be.

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Later, a divisor class was replaced by an equivalent notion, *holomorphic line bundle***.**

There is a more general object called a *holomorphic vector bundle* **on X. This consists of a space E with a map onto X, in which all the fibres have the structure of a vector space which does not vary locally. The dimension of all the E_t is called** *the dimension* **or** *rank* **of E itself. If the rank is 1, it is called a** *line bundle***.**

Vector bundles

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If V is a vector bundle, one has thus, a lot of satellite bundles associated to it. Just as one can talk about the dual of a vector space, the exterior product, etc. one may talk of the dual, etc. of a vector bundle as well. In particular, if its rank is r and the r-th exterior power is a line bundle, and we may call the *determinant* of E.

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It was reslised that isomorphism classes of vector bundles could not be classified in terms of any structure we normally study. It was Mumford who realised the kind of vector bundles that are amenable to a classification. He called them *stable bundles.*

13. Moduli space of vector bundles.

The *slope* **of a vector bundle** E **is the rational number** $\mu(E) = deg(E)/rank(E)$**. A vector bundle is** *stable* **if every sub-bundle has less slope than that of** E**.**

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A vector bundle is *polystable* if it is a direct sum of stable bundles, all of the same slope. In any family of bundles, those which are stable, form an open set.

Polystable bundles are the ones which allow classication. Mumford proved that isomorphism classes of stable bundles of rank r and degree d are classified by a smooth variety of dimension $r^2(g-1)+1$. Seshadri showed that those of polystable bundles form a projective variety $U(r,d)$.

14. Unitary representations of the fundamental group

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Given X of genus g, one thus gets, similar to the Jacobian, a host of varieties $U(r, d)$. Narasimhan and I began a study of these varieties and occupied ourselves for many years on this question.

Early results

Since there is a natural map $det : (U(r, d) \to J^d$, the study can be reduced to isomorphism clssses of polystable bundles whose determinant is a fixed line bundle ξ. We will denote it by $SU(r, \xi)$.

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Another early result is that when $g = 2$, $SU(2,\xi)$ is a 3-dimensional projective space if the degree is even, and the intersection of two quadrics in P^5 if the degree is odd.

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Hitchin provided the answer. Consider pairs consistng of a vector bundle E and a linear map $h : E \to E \otimes T^*$ where T is the tangent bundle of X. One can define polystable pairs in this case similar to the above. The moduli of such pairs and the isomorphism classes of representations have the same classification!

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Surprise! Surprise! This is something sought after by physicists.

THANK YOU!